Superconducting Sphere in an External Magnetic Field Revisited

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Abstract
The purpose of this article is to give the intelligible procedure for undergraduate students to grasp proof of the fact that the magnetic field outside the hollow superconducting sphere (superconducting shell) coincides with the field of a point magnetic dipole both when an uniform external magnetic field is applied as when a ferromagnetic sphere is located at the shell's geometric center. The proof is based on the London's equation transformation to an integral equation for the vector potential of system. The integral equation may be solved completely in case of the solid sphere.
Keywords: Meissner effect, London's equation, vector potential, integral equation.

Introduction

The problem of uniform magnetic field expulsion, it's value B_{ext} is less then the critical one (B_{cr}), from the superconducting solid sphere volume (figure 1) was discussed in textbooks on the electrodynamics and on the solid-state physics many times. For a student – physicist it is the useful illustration of Meissner effect in elementary functions and without any idealization of situation, which is occurred, for example, in the problem of "infinite" slab in a parallel field. For the case of approximation $\lambda \to 0$, where λ is the field penetration depth in a superconductor we find it's presentation in the initial university physics course already (Sivouchine, 1983). By this author of the textbook makes educated guess, that in an external field the sphere behaves like a point magnetic dipole with an unknown magnetic moment $\mathfrak{R}$. Using of such educated guess seems to be quite acceptable in the general physics course however it is hardly right in textbooks for undergraduate students. Meanwhile the prompt message that the supercurrent modulus $j(\boldsymbol{R})$ angular dependence is

$$j(\boldsymbol{R}) = j_0(R) \cdot \sin\theta, \tag{1}$$

(R, θ are the coordinates shown in figure 1, j_0 is a function of only the R coordinate) may be occured, for example, in (Batygin and Toptygin, 1978) although this book is intended for a more advanced student. In this textbook the superconducting solid sphere is considered out of the approximation $\lambda \to 0$ already, but in the situation when the London's equation

$$\text{rot}\,\boldsymbol{j} = -\frac{c}{4\pi\lambda^2}\boldsymbol{B} \tag{2}$$

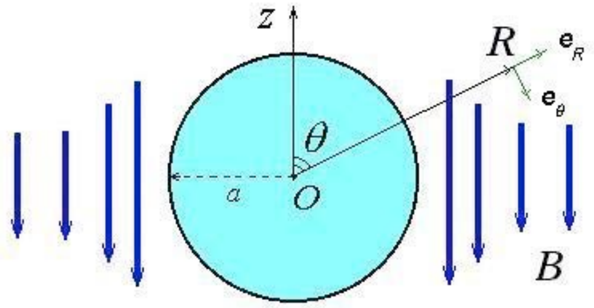

Fig. 1. *Solid superconducting sphere of radius a in an external uniform magnetic field. The arrows are the total magnetic field **B** induction vectors in different points of the sphere's equatorial plane*

is applicable ($\boldsymbol{B}(\boldsymbol{R})$ is the total magnetic induction vector, $\boldsymbol{B} = \boldsymbol{B}_{\text{ext}} + \boldsymbol{B}_{\text{j}}$, where $\boldsymbol{B}_{\text{j}}(\boldsymbol{R})$ is the supercurrent field induction vector, c is the velocity of light). The method of solution in (Batygin and Toptygin, 1978) is the next: the projection of equation for supercurrent,

$$\text{rot rot}\,\boldsymbol{j} = -\frac{1}{\lambda^2}\cdot\boldsymbol{j} \tag{3}$$

on the $\boldsymbol{e}_\varphi$ basis vector of the figure 1 coordinate system, is, taking into account the supercurrent modulus independence on the azimuthal angle φ:

$$\Delta j - \frac{1}{t^2\sin\theta}j = \frac{\partial^2 j}{\partial t^2} + \frac{2}{t}\frac{\partial j}{\partial t} + \frac{1}{t^2}\frac{\partial^2 j}{\partial\theta^2} + \frac{1}{t^2}\frac{\cos\theta}{\sin\theta}\frac{\partial j}{\partial\theta} - \frac{1}{t^2\sin\theta}\cdot j = j, \tag{4}$$

($t = R/\lambda$). Equation (1) yields with (4)

$$t^2\cdot j_0'' + 2t\cdot j_0' - (t^2+2)\cdot j_0 = 0. \tag{5}$$

The general solution of (5) is (Nikiforov and Uvarov, 1988):

$$j_0(t) = k_1\left(\frac{1}{t} - \frac{1}{t^2}\right)e^t + k_2\left(\frac{1}{t} + \frac{1}{t^2}\right)e^{-t}, \tag{6}$$

where k_1 and k_2 are some constants being defined by boundary conditions. For the solid sphere we must demand $\boldsymbol{j}(\boldsymbol{R} = 0) = \boldsymbol{0}$. Expanding exponents in (6) into a series and inserting $t = 0$ there, we obtain $k_1 = k_2 = k$. Further, using (2), student may calculate the total field induction value $\boldsymbol{B}(\boldsymbol{R})$ at $R \le a$ and, in particular, the value $B(a-0)$. Postulating that at $R > a$

$$B_r = \left(B_{\text{ext}} + \frac{2\cdot\mathfrak{R}}{R^3}\right)\cdot\cos\theta, \tag{7}$$

$$B_\theta = \left(-B_{\text{ext}} + \frac{\mathfrak{R}}{R^3}\right)\cdot\sin\theta, \tag{8}$$

and joining components of $\boldsymbol{B}$ at $R = a - 0$ and at $R = a + 0$ one can derive

$$k = 3c \cdot B_{\text{ext}} a / 16\pi\lambda^2 \sinh(a/\lambda), \tag{9}$$

$$\Re = \frac{B_{\text{ext}} \cdot a^3}{2} \cdot \left\{ 3 \cdot \left(\frac{\lambda}{a \cdot \tanh(a/\lambda)} - \left(\frac{\lambda}{a} \right)^2 \right) - 1 \right\}, \tag{10}$$

and the problem is solved completely.
The method of attack of Batygin and Toptygin was practically criticized in the article (Matute, 1999). On author's opinion, «the general formal solution is avoided» in (Batygin and Toptygin, 1978) and other textbooks, although «the problem can be solved by standard methods which is actually presented in the textbook». In the article (Matute, 1999) the field equation

$$\text{rot rot}\boldsymbol{B} = -\frac{1}{\lambda^2} \cdot \boldsymbol{B}, \quad R \le a, \tag{11}$$

is considered together with the Maxwell equations. Combining it and taking into account the symmetry of problem one come to some equation for the B_r field component only. Solution of this equation is searched as a series in the Legendre's polynomials $P_n(\cos\theta)$. The B_θ component of vector $\boldsymbol{B}$ is searched as a series in terms of the functions $P^1{}_n = dP_n(\cos\theta)/d\theta$.
The Maxwell equations solution for $R > a$ is known (Jackson, 1962):

$$B_r = B_{\text{ext}} \cdot \cos\theta + \sum_{n=0}^{n=\infty} \frac{c_n}{t^{n+2}} P_n(\cos\theta), \tag{12}$$

$$B_\theta = -B_{\text{ext}} \cdot \sin\theta - \sum_{n=0}^{n=\infty} \frac{c_n}{(n+1) \cdot t^{n+2}} \cdot P_n^{'}(\cos\theta), \tag{13}$$

where c_n are still unknown coefficients. Equating (12) – (13) and the solution of (11) at $r = a$, they derive c_n, by this $c_n = 0$ for all indexes $n > 1$. That is to say that when $R > a$, the field $\boldsymbol{B}_{\text{j}}$ of superconducting solid sphere coincide with the field of a point dipole. Thus the formulated problem may be solved without any prior guess. The results of article (Matute, 1999) gave the groundwork, which is necessary for the exhaustive understanding of the solution of problem by an undergraduate student. However attempts to adapt the algorithm, having been developed in this work to analogous problem concerning the hollow sphere show noticeable increasing of the routine calculations volume, which are necessary for the proof of the super-current field $\boldsymbol{B}_{\text{j}}$ dipolar character. This proof became even more cumbersome when the magnetic dipole is added into the sphere's cavity. Can an educator recommend to an undergraduate student the other way of solution, which will be not more cumbersome and will be interesting for the intellect?

Hollow superconducting sphere (superconducting shell) in an uniform magnetic field.

Let us consider the hollow superconducting sphere of inner radius h, outer radius a and volume V (figure 2). Let us prove the next prime statement: in the external uniform field $\boldsymbol{B}_{\text{ext}}$

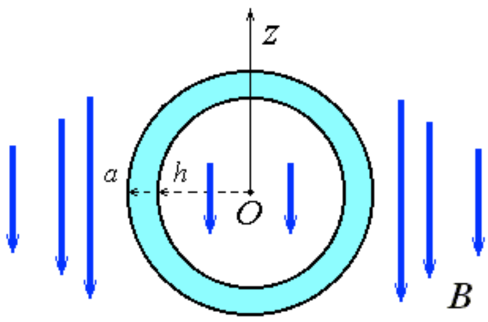

Fig. 2. *Hollow superconducting sphere of inner radius h and outer radius a in an uniform magnetic field. The context of arrows is the same as in the previous figure.*

the field of supercurrent $\boldsymbol{B}_j(R > a)$ coincides with the field of a point dipole. For this purpose we introduce vector potentials $\boldsymbol{A}_j$, $\boldsymbol{A}_{ext}$, $\boldsymbol{A}$ accordingly the formulae $\boldsymbol{B}_j = \mathrm{rot}\boldsymbol{A}_j$, $\boldsymbol{B}_{ext} = \mathrm{rot}\,\boldsymbol{A}_{ext}$, $\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}$. Owing to the cylindrical symmetry of problem we can calibrate it so that all of the three vector potential modulus are not depend on the azimuthal angle φ of our coordinate system. Then all of the three vector potentials divergences are zero in all space. According to (Batygin and Toptygin, 1978) in such a situation

$$\boldsymbol{A}(\boldsymbol{R}) = \boldsymbol{A}_{ext}(\boldsymbol{R}) + \boldsymbol{A}_j(\boldsymbol{R}). \tag{14}$$

By this (Jackson, 1962; Batygin and Toptygin, 1978),

$$\boldsymbol{A}_j(\boldsymbol{R}) = \frac{1}{c} \cdot \int_V \frac{\boldsymbol{j}(\boldsymbol{R}')}{|\boldsymbol{R} - \boldsymbol{R}'|} dV'. \tag{15}$$

As div$\boldsymbol{A} = 0$, one can use the London's equation in the form

$$\boldsymbol{j}(\boldsymbol{R}) = -\frac{c}{4\pi\lambda^2} \cdot \boldsymbol{A}(\boldsymbol{R}). \tag{16}$$

Combining (14) – (16), we derive ($h \le R \le a$):

$$\boldsymbol{A}(\boldsymbol{R}) = \boldsymbol{A}_{ext}(\boldsymbol{R}) - \frac{1}{4\pi\lambda^2} \cdot \int_V \frac{\boldsymbol{A}(\boldsymbol{R}')}{|\boldsymbol{R} - \boldsymbol{R}'|} dV', \tag{17}$$

where

$$\boldsymbol{A}_{ext}(\boldsymbol{R}) = \frac{B_{ext} \cdot R}{2} \sin\theta \cdot \boldsymbol{e}_\varphi . \tag{18}$$

Let us confine ourself to $\boldsymbol{R}$ radius-vector, lying in the plane of figures 1 – 2. Projecting the equation (17) on the axis, which lies perpendicularly to this plane (Jackson, 1962), we derive the equation for the vector potential modulus already

$$A(\boldsymbol{R}) = \frac{B_{\text{ext}} \cdot R}{2}\sin\theta - \frac{1}{4\pi\lambda^2}\cdot\int_V \frac{A(\boldsymbol{R}')}{|\boldsymbol{R}-\boldsymbol{R}'|}\cdot\cos\varphi' dV'. \tag{19}$$

In dimensionless variables $\boldsymbol{r} = \boldsymbol{R} / a$, $\widetilde{\boldsymbol{A}} = \dfrac{2}{B_{\text{ext}}a}\cdot\boldsymbol{A}$, this equation may be rewritten as

$$\widetilde{A}(\boldsymbol{r}) = r\cdot\sin\theta - \delta\cdot\int_{\widetilde{V}} \frac{\widetilde{A}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\cos\varphi' dV'. \tag{20}$$

Here $\delta = a^2/4\pi\lambda^2$, $\widetilde{V}$ is the volume of unit outer radius hollow sphere in an imaginary space; it's inner radius is h / a. We have the so named 2 – type nonhomogeneous Fredholm equation. One of theorems of the integral equations theory states (Korn and Korn, 1968): if the integral $F(\boldsymbol{r}) = \int_{\widetilde{V}} (\boldsymbol{r}-\boldsymbol{r}')^{-2} dV'$ is bounded function in $\widetilde{V}$, then the number ξ exists such that at all $\delta < \xi$, Neumann's step-by-step approximations series

$$\widetilde{A}^{[1]}(\boldsymbol{r}) = r\cdot\sin\theta, \tag{21}$$

$$\widetilde{A}^{[n+1]}(\boldsymbol{r}) = r\cdot\sin\theta - \delta\cdot\int_{\widetilde{V}} \frac{\widetilde{A}^{[n]}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\cos\varphi' dV', \tag{22}$$

is converging to a solution of (20) uniformly inside $\widetilde{V}$. One can find in (Prudnikov, Brychkov and Marichev, 1986):

$$G(\boldsymbol{r}) = \int_{r'^2 \le 1} \frac{dV'}{(\boldsymbol{r}-\boldsymbol{r}')^2} = 2\pi\int_{\alpha=0}^{\infty} \frac{d\alpha}{(1+\alpha)^2}\left(1+(1-r^2)\alpha\right)^{1/2} \tag{23}$$

and the boundedness of both function $G(\boldsymbol{r})$ as a fortiori function $F(\boldsymbol{r}) < G(\boldsymbol{r})$ at $0 < r \le 1$ became evident. Suppose the formula $\lambda = \lambda(0)/\sqrt{1-(T/T_{\text{cr}})^4}$ is correct (Ashkroft and Mermin, 1976) where T_{cr} is the superconducting transition temperature. We conclude that some temperature interval $T_x \le T \le T_{\text{cr}}$ exists where the series of functions (21) – (22) is converging to a solution of equation (20). Near the T_{cr}, where $\delta \to 0$:

$$\widetilde{A}(r) \approx \widetilde{A}^{[2]}(\boldsymbol{r}) = \widetilde{A}^{[1]}(\boldsymbol{r}) + \Delta\widetilde{A}(\boldsymbol{r}) = r\cdot\sin\theta - \delta\cdot\int_{\widetilde{V}} \frac{r'\cdot\sin\theta'}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\cos\varphi' dV'. \tag{24}$$

Using the expansion of $1/|\boldsymbol{r}-\boldsymbol{r}'|$ in spherical harmonics (Jackson, 1962) we obtain

$$\Delta\widetilde{A}(r) = -4\pi\delta\cdot\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\mathrm{Y}_{l,m}(\theta,\varphi=0)\cdot\int_{r'=h/a}^{1} F_l(r,r')r'^3 dr'\int_S \mathrm{Y}_{l,m}(\theta',\varphi')\cdot\sin\theta'\cos\varphi' dS' \tag{25}$$

, where S is the unit sphere surface, $F_l(r,r') = \min(r,r')^l / \max(r,r')^{l+1}$. As

$$\sum_{m=-l}^{l} \mathrm{Y}_{l,m}(\theta,\varphi=0)\cdot\int_S \mathrm{Y}_{l,m}(\theta',\varphi')\sin\theta'\cdot\cos\varphi' dS' = \delta_{l,1}\cdot\sin\theta, \tag{26}$$

what is verified easy ($\delta_{m,\,n}$ is the Kronecker symbol), then

$$\widetilde{A}^{[2]}(\boldsymbol{r}) = \widetilde{f}_2(r)\cdot\sin\theta, \tag{27}$$

where $\widetilde{f}_2(r) = r - \frac{4\pi}{3}\delta\cdot\int\limits_{r'=h/a}^{1} F_1(r,r')r'^3 dr'$ is a function only of the r variable (there is no necessity to calculate $\widetilde{f}_2(r)$ in an explicit form although it may be made very easy). It follows from (22), (26) and (27) that formula with the structure analogical to (27) will be reproduced at any step of iteration process with character replacement 2 → n in (27). So, when the superconductor is cooled down from T_{cr} temperature, only the radial distribution of field is changed but it's angular distribution is unchanged. We conclude that in all the region of Neumann's series (20) – (21) convergence, at $h \le R \le a$,

$$\boldsymbol{A}(\boldsymbol{R}) = f(R)\cdot\sin\theta\cdot\boldsymbol{e}_\varphi. \tag{28}$$

where $f(R)$ is a function only of the R variable. Let us insert (28) and (16) in (15) at $R > a$. It gives:

$$\boldsymbol{A}_{\mathrm{j}}(R>a) = -\frac{B_{\mathrm{ext}}a}{R^2\cdot 6\lambda^2}\left(\int_h^a f(t)t^3 dt\right)\cdot\sin\theta\cdot\boldsymbol{e}_\varphi. \tag{29}$$

This is the vector potential of a point magnetic dipole. We have proved the prime statement at the temperatures $T_x \le T \le T_{\mathrm{cr}}$. That will do as we now have a right to use the function (6) in this temperature interval and further to confirm that formula (6) is correct at an arbitrary temperature.
Now if a student knows indefinite integrals of the elementary functions, he will calculate the value $\boldsymbol{A}(\boldsymbol{R})$ at $h \le R \le a$ explicitly without problems. For this purpose it's necessary to insert (1), (6) and (16) in the both parts of (20) and to equate coefficients at the same degrees of r. We present the final result:

$$f(R) = \frac{3B_{\mathrm{ext}}a}{2S}\cdot\left[\left(\frac{\lambda}{R}-\frac{\lambda^2}{R^2}\right)\cdot U\cdot e^{\frac{R-h}{\lambda}} + \left(\frac{\lambda}{R}+\frac{\lambda^2}{R^2}\right)\cdot W\cdot e^{\frac{h-R}{\lambda}}\right], \tag{30}$$

where $U = \frac{h^2}{3\lambda^2}+\frac{h}{\lambda}+1$, $W = \frac{h^2}{3\lambda^2}-\frac{h}{\lambda}+1$ and

$$S = U\cdot e^{\frac{a-h}{\lambda}} - W\cdot e^{\frac{h-a}{\lambda}}. \tag{31}$$

Inside the cavity, the formulae (15) – (16) give:

$$\boldsymbol{A}_{\mathrm{j}}(R<h) = -\frac{R}{3\lambda^2}\left(\int_h^a f(t)dt\right)\cdot\sin\theta\cdot\boldsymbol{e}_\varphi, \tag{32}$$

and after a simple calculations

$$\boldsymbol{B}(R < h) = 2a \cdot \boldsymbol{B}_{\text{ext}} / \lambda S. \tag{33}$$

Expanding exponents in (31) as a series in the parameter $\varepsilon = a / \lambda$, one can prove that at any h and a it will be $B(R < h) < B_{\text{ext}}$ always. Hollow superconducting sphere in a field of point dipole located at it's center and the generalization of problems.

In the article (Hurault and Pincus, 1969), the problem was formulated inter alia: to determine magnetic field inside the infinite superconductor surrounding a monodomain ferromagnetic sphere of radius ℓ and of magnetization M_s. The solution of equations (3) and (16) for this

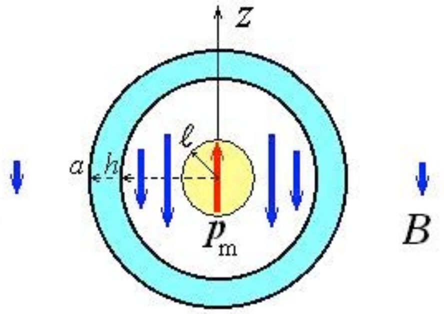

Fig. 3. Hollow superconducting sphere of inner radius h and outer radius a in the field of a magnetized sphere with the dipolar moment $\boldsymbol{p}_{\text{m}}$. *The context of arrows is the same as in the previous figures.*

problem, resulting in this article is

$$\boldsymbol{A}(R \geq \ell) = 4\pi M_s \frac{\ell^3}{R^2} \frac{R/\lambda + 1}{3 + 3\ell/\lambda + \ell^2/\lambda^2} e^{-(R-\ell)/\lambda} \sin\theta \cdot \boldsymbol{e}_\varphi. \tag{34}$$

The method being developed in the previous paragraph allows to provide the necessary background for this formula which is absent in (Hurault and Pincus, 1969). We will make it for the more common situation of the superconducting shell in the field of a ferromagnetic sphere with the magnetic moment p_{m} located in it's center (figure 3). Just the same method as in the previous paragraph leads us to the equation,

$$\boldsymbol{A}(\boldsymbol{R}) = \frac{p_{\text{m}}}{R^2} \sin\theta \cdot \boldsymbol{e}_\varphi - \frac{1}{4\pi\lambda^2} \cdot \int_V \frac{\boldsymbol{A}(\boldsymbol{R}')}{|\boldsymbol{R} - \boldsymbol{R}'|} dV' \tag{35}$$

instead of the equation (19). The dimensionless variables are here $\boldsymbol{r} = \boldsymbol{R} / a$, $\widetilde{\boldsymbol{A}} = a^2 \cdot \boldsymbol{A} / p_{\text{m}}$. Then

$$\widetilde{A}(\boldsymbol{r}) = \frac{1}{r^2} \cdot \sin\theta - \delta \cdot \int_V \frac{\widetilde{A}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \cdot \cos\varphi' dV'. \tag{36}$$

Further consideration duplicates the text beginning from formula (22) to formula (28) with the rewritting of function $\widetilde{A}^{[1]}(\boldsymbol{r}) = r^{-2} \cdot \sin\theta$ instead of function $\widetilde{A}^{[1]}(\boldsymbol{r}) = r \cdot \sin\theta$ overall. Inserting (1), (6) and (16) to the both parts of (36) and equating coefficients of the same degrees of r, we obtain

$$f(R) = \frac{p_{\mathrm{m}}}{\lambda^2 \cdot S} \cdot \left[\left(\frac{\lambda}{R} - \frac{\lambda^2}{R^2} \right) \cdot e^{\frac{R-a}{\lambda}} + \left(\frac{\lambda}{R} + \frac{\lambda^2}{R^2} \right) \cdot e^{\frac{a-R}{\lambda}} \right]. \tag{37}$$

When $a \to \infty$ and $\ell = h$ we reproduce the result of (Hurault and Pincus, 1969).
The field inside the cavity ($\ell < R < h$):

$$\boldsymbol{B}(\boldsymbol{R}) = \boldsymbol{B}_{\mathrm{fm}}(\boldsymbol{R}) - \frac{4}{3} \cdot \frac{\boldsymbol{p}_{\mathrm{m}} \cdot \sinh\left(\frac{a-h}{\lambda} \right)}{h\lambda^2 \cdot S}. \tag{38}$$

where $\boldsymbol{B}_{\mathrm{fm}}(\boldsymbol{R})$ is the field of ferromagnet outside it.

Let $\boldsymbol{A}_1(\boldsymbol{R})$ is the solution of the equation (17) for the superconducting shell in the field of the uniform magnetic field B_{ext} and $\boldsymbol{A}_2(\boldsymbol{R})$ is the one's solution for just the same shell in the field of point dipol being located in the center of shell ($\boldsymbol{B}_1 = \mathrm{rot}\boldsymbol{A}_1$, $\boldsymbol{B}_2 = \mathrm{rot}\boldsymbol{A}_2$, $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$). For the case of very weak field $\boldsymbol{B}$ ($B < B_{\mathrm{cr}}$ is elsewhere), $\boldsymbol{A}(\boldsymbol{R}) = \boldsymbol{A}_1(\boldsymbol{R}) + \boldsymbol{A}_2(\boldsymbol{R})$ will be the vector-potential for just the same superconductor in the field both of that dipol inside it and of B_{ext} outside it. A consequence of the results stated above is the fact that the formula (28) will be correct also in this most common situation. The detailed calculations of the field distribution may be easy executed with the help of formulae (30) and (37).

Conclusions

Using results of the integral equations theory allows to put the novel method of solution of the problem about the superconducting sphere in a magnetic field into pedagogical practice. This method is alternative to the traditional one, when the London's equation solution inside a superconductor and the Maxwell's equations solution outside it are joined on it's surface. The advantage of this method over the tradition one lies in the fact that the change from the problem about the solid sphere in an uniform field (problem №1) to the problem about the hollow sphere in an uniform field plus the field of a dipole (problem №2) doesn't bring the solution's significant amplification. One may say that now not only the problem №1 is open to understanding of a student but also the problem №2. Nevertheless if an educator on the lecture wants to confine oneself to the situation of the solid sphere, he can now demonstrate the solution of equation (20) which is given in Appendix instead of to integrate the contents of (Matute, 1999) and (Batygin and Toptygin, 1978).

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Appendix

When h = 0, the integral equation (20) is solved very simply and there is no necessity to cite the formula (6). For this purpose let us insert (28) in (19). In combination with (26) it gives the next integral equation for the function $\omega(r) = 2f(r)/B_{\text{ext}}a$:

$$\omega(r) = r - \frac{\varepsilon^2}{3} \cdot \left\{ \frac{1}{r^2} \cdot \int_0^r \omega(t) \cdot t^3 dt + r \cdot \int_r^1 \omega(t) dt \right\}. \tag{39}$$

As aforesaid, the functional sequence $\omega^{[n]}(r)$, where $\widetilde{A}^{[n]}(\boldsymbol{r}) = \omega^{[n]}(\boldsymbol{r}) \cdot \sin\theta$ convergents to a solution of (39) at $0 \le r \le 1$ and at an infinitesimal ε. Being made some iterations, student may notice that

$$\omega^{[n]}(r) = \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=i} b_{j,i}^n r^{2j-1} \right) \cdot \varepsilon^{2(i-1)} \tag{40}$$

where $b^n_{j,i}$ are some coefficients. Formula (40) is proved elementary by the method of mathematical induction. Suppose $\omega(r)$ is infinitely differentiable function at $r = 0$ both on r and on ε variables. So when $n \to \infty$, coefficients of the series (40) must tend to the coefficients $(1/k! \cdot m!) \cdot \partial^{k+m} \omega / \partial^k \varepsilon \ \partial^m r \big|_{\varepsilon=r=0}$ of the function $\omega(r)$ Taylor's series. It signifies that we may change the sequential order of indexes in (40):

$$\omega^{[n]}(r) = \sum_{j=1}^{j=n} \left(\sum_{i=j}^{i=n} b_{j,i}^n \varepsilon^{2(i-1)} \right) \cdot r^{2j-1}. \tag{41}$$

So we may search the solution of (39) as

$$\omega(r) = \sum_{j=1}^{j=\infty} b_j \cdot r^{2j-1} \tag{42}$$

($b_j = \lim_{n\to\infty} \sum_{i=j}^{i=n} b_{j,i}^n \varepsilon^{2(i-1)}$). Inserting (42) in both parts of (39) and equating coefficients at the same degrees we derive

$$b_1 - 1 + \frac{\varepsilon^2}{3} \cdot \sum_{j=1}^{j=\infty} \frac{b_j}{2j} = 0, \tag{43}$$

$$b_{j+1} = \frac{\varepsilon^2}{2j(2j+3)} \cdot b_j. \tag{44}$$

Now let's insert (44) in (43). It gives

$$b_1 \cdot \left[1 + \frac{\varepsilon^2}{2 \cdot 3} + \frac{\varepsilon^4}{2 \cdot 4 \cdot 3 \cdot 5} + \frac{\varepsilon^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 5 \cdot 7} + \ldots \right] = 1, \tag{45}$$

so that $b_1 = \varepsilon / \sinh\varepsilon$ and consequently

$$\omega(r) = b_1 \cdot \left(r + \frac{\varepsilon^2 \cdot r^3}{2 \cdot 5} + \frac{\varepsilon^4 \cdot r^5}{2 \cdot 4 \cdot 5 \cdot 7} + \frac{\varepsilon^6 \cdot r^7}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 9} + \ldots \right) =$$

$$= \frac{3b_1}{\varepsilon} \cdot \left(\frac{1}{3} \varepsilon \cdot r + \frac{4}{5!} (\varepsilon \cdot r)^3 + \frac{6}{7!} (\varepsilon \cdot r)^5 + \frac{8}{9!} (\varepsilon \cdot r)^7 + \ldots \right) = \frac{3}{\sinh\varepsilon} \cdot \left(\frac{\cosh(\varepsilon \cdot r)}{\varepsilon \cdot r} - \frac{\sinh(\varepsilon \cdot r)}{(\varepsilon \cdot r)^2} \right). \tag{46}$$

Definitely

$$\boldsymbol{A}(R \le a) = \frac{3B_{\text{ext}} a}{2\sinh(a/\lambda)} \left(\frac{\cosh(R/\lambda)}{R/\lambda} - \frac{\sinh(R/\lambda)}{(R/\lambda)^2} \right) \sin\theta \cdot \boldsymbol{e}_\varphi. \tag{47}$$

Inserting (46) to (29) we immediately come to (10).