Basis Vectors in Relativity

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Abstract

Basis vectors $\boldsymbol{e}_\alpha$ play a useful role in special and general relativity. In particular they allow an expansion of the vectorial spacetime interval $d\boldsymbol{l}$ along infinitesimal curvilinear coordinate differences $d\boldsymbol{l} = \boldsymbol{e}_\alpha d\xi^\alpha$: (thus the definition $\boldsymbol{e}_\alpha = d\boldsymbol{l} / d\xi^\alpha$) In this work, expressions for basis vectors obtained directly from the covariant metric coefficients are given and discussed. Forms for nonorthogonal coordinate systems are either unknown or hard to find in the literature; an application to a nonorthogonal coordinate system is therefore worked out.

Keywords: Relativity, basis vectors, Minkowski space

INTRODUCTION

It suffices for most situations in the theories of relativity to assume that spacetime is a pseudo-Riemannian manifold, i.e. one in which at each point of spacetime a *local inertial frame* (LIF) with Cartesian coordinates $\boldsymbol{x} = (x^0, x^1, x^2, x^3)$ can be defined (e.g. a Minkowski fourspace). Although much of the ensuing material is discussed in many texts (Bergmann, 1976; Hartle, 2003; Hobson, Efstathiou, & Lasenby, 2006; Misner, Thorne, & Wheeler, 1984; Schutz, 2009), there are issues, especially for nonorthogonal coordinate systems, that require some discussion, among others, for the benefit of students. Also missing in most, if not all texts, are expressions for the contravariant basis vectors.

It is also assumed that the manifold can be described by a metric given in a curvilinear coordinate system $\boldsymbol{x}$ with line element $d\boldsymbol{l}$ such that

$$(d\boldsymbol{l})^2 = \sum_\alpha g_{\alpha\beta}(\boldsymbol{\xi}) d\xi^\alpha d\xi^\beta \tag{1}$$

There then are two ways of describing the line element:

$$d\boldsymbol{l} = \sum_{\alpha} dl^{\alpha} \hat{\boldsymbol{x}}_{\alpha} = \sum_{\alpha} \boldsymbol{e}_{\alpha} d\xi^{\alpha} = \sum_{\alpha} \boldsymbol{e}^{\alpha} d\xi_{\alpha} \tag{2}$$

The first expresses $d\boldsymbol{l}$ in terms of the directional unit vectors $\hat{\boldsymbol{x}}_{\alpha}$, the second in terms of *covariant basis vectors* $\boldsymbol{e}_{\alpha}$ and the third in terms of *contravariant basis vectors* $\boldsymbol{e}^{\alpha}$. The covariant coordinate increment is $d\xi_{\alpha} = g_{\alpha\beta} d\xi^{\beta}$ in which expression the Einstein summation convention has been assumed (as it will be henceforth unless there is ambiguity in the notation). Thus $\boldsymbol{e}_{\alpha}$ is a vector along the tangent to the curve with increment $d\boldsymbol{l}^{\alpha}$. The definitions (2) also lead to

$$\boldsymbol{e}_{\alpha} = \frac{d\boldsymbol{l}}{d\xi^{\alpha}} = \hat{\boldsymbol{x}}_{\beta} \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \tag{3}$$

in terms of *local inertial frame* (LIF) coordinates at a location *P* in the manifold. If the transformation between the $\boldsymbol{x}$ and $\boldsymbol{x}$ coordinates are known, then this is a useful form for the covariant basis vector. But if not, then one finds via (1) and (2) that

$$\boldsymbol{e}_{\alpha} \cdot \boldsymbol{e}_{\beta} = g_{\alpha\beta} \tag{4}$$

which yields $|\boldsymbol{e}_{\alpha}| = \sqrt{|g_{\alpha\alpha}|}$. Note that the absolute value of the metric coefficient is needed; this is because in a number of important models $g_{00} < 0$. We will clarify below how this is taken into account. The covariant basis vector $\boldsymbol{e}_{\alpha}$ being a tangent vector along $\hat{\boldsymbol{x}}_{\alpha}$, then becomes (no summation)

$$\boldsymbol{e}_{\alpha} = \hat{\boldsymbol{x}}_{\alpha} \sqrt{|g_{\alpha\alpha}|} \tag{5}$$

This, together with (4) yields

$$g_{\alpha\beta} = \sqrt{|g_{\alpha\alpha} g_{\beta\beta}|} (\hat{\boldsymbol{x}}_{\alpha} \cdot \hat{\boldsymbol{x}}_{\beta}) \tag{6}$$

Evidently $g_{\alpha\beta}=0$ when $\hat{\boldsymbol{x}}_\alpha \cdot \hat{\boldsymbol{x}}_\beta = 0$, i.e. for orthogonal pairs of unit vectors. Furthermore $g_{\alpha\beta}<0$ implies $\hat{\boldsymbol{x}}_0 \cdot \hat{\boldsymbol{x}}_0 < 0$ as is the case in a hyperbolic Minkowski space.[1 2]

It is a bit more work to obtain expressions for contravariant basis vectors. These follow from $d\xi_\alpha = g_{\alpha\beta} d\xi^\beta$ (lowering an index using the metric coefficient), and consequently $\boldsymbol{e}_\alpha = \dfrac{d\boldsymbol{l}}{d\xi^\gamma} = \sum_{\alpha,\beta} \boldsymbol{e}^\alpha g_{\alpha\beta} \dfrac{\partial \xi^\beta}{\partial \xi^\gamma} = \sum_\alpha \boldsymbol{e}^\alpha g_{\alpha\beta}$ which then gives rise to

$$\boldsymbol{e}_\alpha = \frac{d\boldsymbol{l}}{d\xi^\gamma} = \sum_{\alpha,\beta} \boldsymbol{e}^\alpha g_{\alpha\beta} \frac{\partial \xi^\beta}{\partial \xi^\gamma} = \sum_\alpha \boldsymbol{e}^\alpha g_{\alpha\gamma} \tag{7}$$

In 4-D Minkowski space this is explicitly a set of 4 linear equations in the vectors (where indices 1,2,3,4 are used instead of 1,2,3,0 as is more usual in relativity):

$$\boldsymbol{e}_1 = \boldsymbol{e}^i \sum_{i=1}^4 g_{i1}, \quad \boldsymbol{e}_2 = \boldsymbol{e}^i \sum_{i=1}^4 g_{i2}, \quad \boldsymbol{e}_3 = \boldsymbol{e}^i \sum_{i=1}^4 g_{i3}, \quad \boldsymbol{e}_4 = \boldsymbol{e}^i \sum_{i=1}^4 g_{i4} \tag{8a}$$

The solution for $\boldsymbol{e}^1$ in terms of Cramèr determinants is

$$\boldsymbol{e}^1 = \frac{1}{|g|} \begin{vmatrix} \boldsymbol{e}_1 & g_{21} & g_{31} & g_{41} \\ \boldsymbol{e}_2 & g_{22} & g_{32} & g_{42} \\ \boldsymbol{e}_3 & g_{23} & g_{33} & g_{43} \\ \boldsymbol{e}_4 & g_{24} & g_{34} & g_{244} \end{vmatrix} \tag{9a}$$

where $|g|$ is the determinant of the covariant metric matrix. If there are no off-diagonal metric coefficients then this reduces to

$$\boldsymbol{e}^1 = \frac{1}{|g|} g_{22} g_{33} g_{44} \boldsymbol{e}_1 = \frac{1}{g_{11}} \boldsymbol{e}_1 = \frac{1}{\sqrt{|g_{11}|}} \hat{\boldsymbol{x}}_1 \tag{9b}$$

The three other contravariant vectors are found similarly. An alternative form is found from two expressions for a line element:

[1] https://*physics.stackexchange.com/questions/290049/are-basis-vectors-imaginary-in-special-relativity*. This reference suggests that 'basis vectors' should be real.

[2] For example, the Schwarzschild metric where $g_{00}(r) = -\left(1-2GM/c^2 r\right)$ in terms of gravitational constant *G*, mass *M*, and radial distance *r*.

$$d\boldsymbol{l} = \sum_{\alpha} \boldsymbol{e}_{\alpha} d\xi^{\alpha} = \sum_{\alpha} (\boldsymbol{\xi}_{\alpha} \sqrt{|g_{\alpha\alpha}|} d\xi^{\alpha})$$
$$d\boldsymbol{l} = \sum_{\alpha} \boldsymbol{e}^{\alpha} d\xi_{\alpha} = \sum_{\alpha,\beta} \boldsymbol{e}^{\alpha} g_{\alpha\beta} d\xi^{\beta} = \sum_{\alpha,\beta} \boldsymbol{e}^{\beta} g_{\beta\alpha} d\xi^{\alpha} = \sum_{\alpha}\sum_{\beta} (g_{\alpha\beta} d\xi^{\beta}) d\xi^{\alpha} \tag{10}$$

Upon equating these two forms one obtains

$$\hat{\boldsymbol{x}}_{\alpha} = \sum_{\beta} \frac{g_{\alpha\beta}}{\sqrt{|g_{\alpha\alpha}|}} \boldsymbol{e}^{\beta} = \sum_{\beta} \frac{\sqrt{|g_{\alpha\alpha} g_{\beta\beta}|}}{\sqrt{|g_{\alpha\alpha}|}} (\hat{\boldsymbol{x}}_{\alpha} \cdot \hat{\boldsymbol{x}}_{\beta}) \boldsymbol{e}^{\beta} = \sum_{\beta} \sqrt{|g_{\beta\beta}|} (\hat{\boldsymbol{x}}_{\alpha} \cdot \hat{\boldsymbol{x}}_{\beta}) \boldsymbol{e}^{\beta} \tag{11}$$

If these equations are solved in four dimensions $\boldsymbol{x} = (\xi^{\alpha}, \xi^{\beta}, \xi^{\gamma}, \xi^{\delta})$ and the indices $\alpha, \beta, \gamma, \delta$ can be 0,1,2,3 in any cyclic permutation, one obtains in determinant form

$$\boldsymbol{e}^{\alpha}(\boldsymbol{x}) = \frac{1}{\sqrt{|g_{\alpha\alpha}|}} \frac{\begin{vmatrix} \hat{\boldsymbol{x}}_{\alpha} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\gamma} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\beta} & 1 & \hat{\boldsymbol{x}}_{\beta}\cdot\hat{\boldsymbol{x}}_{\gamma} & \hat{\boldsymbol{x}}_{\beta}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\gamma} & \hat{\boldsymbol{x}}_{\gamma}\cdot\hat{\boldsymbol{x}}_{\beta} & 1 & \hat{\boldsymbol{x}}_{\gamma}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\delta} & \hat{\boldsymbol{x}}_{\delta}\cdot\hat{\boldsymbol{x}}_{\beta} & \hat{\boldsymbol{x}}_{\delta}\cdot\hat{\boldsymbol{x}}_{\gamma} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\gamma} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\beta}\cdot\hat{\boldsymbol{x}}_{\alpha} & 1 & \hat{\boldsymbol{x}}_{\beta}\cdot\hat{\boldsymbol{x}}_{\gamma} & \hat{\boldsymbol{x}}_{\beta}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\gamma}\cdot\hat{\boldsymbol{x}}_{\alpha} & \hat{\boldsymbol{x}}_{\gamma}\cdot\hat{\boldsymbol{x}}_{\beta} & 1 & \hat{\boldsymbol{x}}_{\gamma}\cdot\hat{\boldsymbol{x}}_{\delta} \\ \hat{\boldsymbol{x}}_{\delta}\cdot\hat{\boldsymbol{x}}_{\alpha} & \hat{\boldsymbol{x}}_{\delta}\cdot\hat{\boldsymbol{x}}_{\beta} & \hat{\boldsymbol{x}}_{\delta}\cdot\hat{\boldsymbol{x}}_{\gamma} & 1 \end{vmatrix}} \tag{12}$$

This again reduces to the simpler form for a diagonal metric given in (9b). Numerical values for $(\hat{\boldsymbol{x}}_{\alpha} \cdot \hat{\boldsymbol{x}}_{\beta})$, etc., are easily inserted into the determinants when needed. It reduces to the 3-D case if one sets $\boldsymbol{\xi}_{\delta} = 0$ (or if $\hat{\boldsymbol{x}}_{\delta} \perp \hat{\boldsymbol{x}}_{\alpha}, \hat{\boldsymbol{x}}_{\beta}, \hat{\boldsymbol{x}}_{\gamma}$) and to the 2-D case if, additionally, $\hat{\boldsymbol{x}}_{\gamma} = 0$. In the latter case,

$$\boldsymbol{e}^{\alpha}(\boldsymbol{x}) = \frac{1}{\sqrt{|g_{\alpha\alpha}|}} \frac{\begin{vmatrix} \hat{\xi}_{\alpha} & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta} \\ \hat{\boldsymbol{x}}_{\beta} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta} \\ \hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta} & 1 \end{vmatrix}} = \frac{1}{\sqrt{|g_{\alpha\alpha}|}\left[1-\left(\hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta}\right)^{2}\right]} \left[\hat{\boldsymbol{x}}_{\alpha} - \left(\hat{\boldsymbol{x}}_{\alpha}\cdot\hat{\boldsymbol{x}}_{\beta}\right)\hat{\boldsymbol{x}}_{\beta}\right] \tag{13}$$

This reduces to $\boldsymbol{e}^{\alpha}(\boldsymbol{x}) = \hat{\boldsymbol{x}}_{\alpha} / \sqrt{|g_{\alpha\alpha}|}$ when $\hat{\boldsymbol{x}}_{\beta} \perp \hat{\boldsymbol{x}}_{\alpha}$ (i.e. when $g_{\alpha\beta} = 0$ for $\beta \neq \alpha$). Yet another definition of a contravariant basis vector via LIF coordinates is

$$\boldsymbol{e}^{\alpha}(\boldsymbol{x}) = \nabla d\hat{\boldsymbol{x}}_{\alpha} = \hat{\boldsymbol{x}}_{\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} \tag{14}$$

from which it is clear from (3) that it has an inverse relationship to $\boldsymbol{e}_{\beta}(\boldsymbol{x})$:

$$\boldsymbol{e}^{\alpha}(\boldsymbol{x}) \cdot \boldsymbol{e}_{\beta}(\boldsymbol{x}) = \delta^{\alpha}_{\beta} \tag{15}$$

In the remaining part of this work, some examples will be worked out to demonstrate, among other things, the equivalence of the various forms of basis vectors.

Example 1: For 3-D spherical coordinates

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{16}$$

yielding $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2\theta$. It follows from $g^{\alpha\alpha} = 1/g_{\alpha\alpha}$ that $g^{rr} = 1$, $g^{\theta\theta} = 1/r^2$, and $g^{\varphi\varphi} = 1/(r\sin\theta)^2$. The basis vectors in these coordinates, from (5) and (9b) are

$$\boldsymbol{e}_r = \hat{r}, \quad \boldsymbol{e}_{\theta} = r\hat{\theta}, \quad \boldsymbol{e}_{\varphi} = r\sin\theta\hat{\varphi}$$

$$\boldsymbol{e}^r = \hat{r}, \quad \boldsymbol{e}^{\theta} = \frac{1}{r}\hat{\theta}, \quad \boldsymbol{e}^{\varphi} = \frac{1}{r\sin\theta}\hat{\varphi} \tag{17}$$

These basis vectors do form an orthogonal set but $\boldsymbol{e}^{\theta}, \boldsymbol{e}_{\theta}$ and $\boldsymbol{e}^{\varphi}, \boldsymbol{e}_{\varphi}$ are not of unit magnitude. The properties of $\boldsymbol{e}_{\alpha} \cdot \boldsymbol{e}^{\beta}$ are then easily obtained and shown to obey (15).

Example 2: Here is an example in which the basis vectors do not form an orthogonal set. Consider two-dimensional (2-D) coordinates ξ^1 and ξ^2 (with the 'vertical' component at 30^o with what normally would be the y axis) and with Cartesian counterparts $x^1(=x)$ and $y^1(=y)$ (Figure 1).

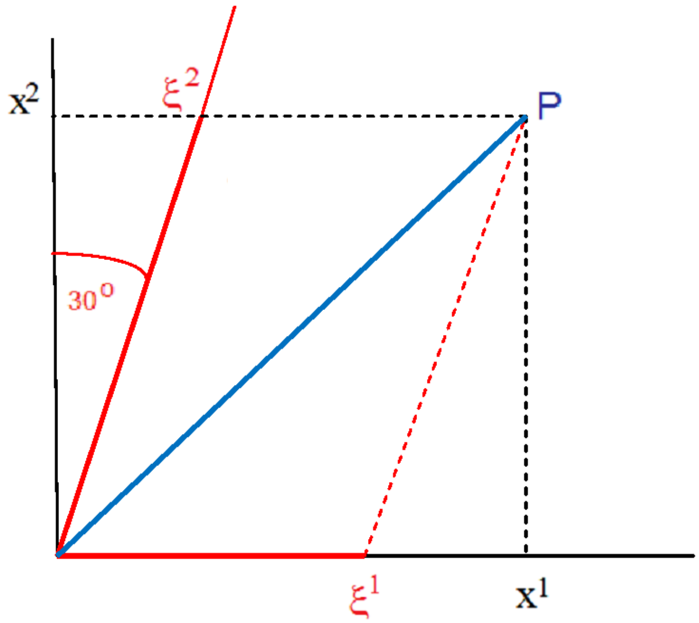

Figure 1. Example 2 figure

It is easily established that

$$\xi^1 = x^1 - \frac{1}{\sqrt{3}} x^2, \quad \xi^2 = \frac{2}{\sqrt{3}} x^2, \quad x^1 = \xi^1 + \frac{1}{2}\xi^2, \quad x^2 = \frac{1}{2}\xi^2\sqrt{3}$$

$$\xi_1 = x_1, \qquad \xi_2 = \frac{1}{2}x_1 + \frac{1}{2}x_2\sqrt{3} \tag{18}$$

which yields $(d\boldsymbol{l})^2 = (dx^1)^2 + (dx^2)^2 = (d\xi^1)^2 + \frac{1}{2}(d\xi^1)(d\xi^2) + \frac{1}{2}(d\xi^2)(d\xi^1) + (d\xi^2)^2$

from which it follows that $g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = g_{21} = \frac{1}{2}$.

Hence from the above:

$$\boldsymbol{e}_1(\boldsymbol{\xi}) = \sqrt{|g_{11}|}\, \xi_1 = \xi_1 = x_1$$

$$\boldsymbol{e}_2(\boldsymbol{\xi}) = \sqrt{|g_{22}|}\, \xi_2 = \xi_2 = \frac{1}{2}(x_1 + x_2\sqrt{3}) \tag{19}$$

in agreement with $g_{ii} = \boldsymbol{e}_i \cdot \boldsymbol{e}_i$. These covariant basis vectors are of unit length, but they are not orthogonal to each other.

For the contravariant case, apply (14): $\boldsymbol{e}^\alpha = \hat{x}_\beta \dfrac{\partial \xi^\alpha}{\partial x^\beta}$ to obtain

$$\begin{aligned} \boldsymbol{e}^1 &= \hat{x}_1 \frac{\partial \xi^1}{\partial x^1} + \hat{x}_2 \frac{\partial \xi^1}{\partial x^2} \quad \rightarrow \quad \boldsymbol{e}^1 = \hat{x}_1 - \frac{1}{\sqrt{3}} \hat{x}_2 \\ \boldsymbol{e}^2 &= \hat{x}_1 \frac{\partial \xi^2}{\partial x^1} + \hat{x}_2 \frac{\partial \xi^2}{\partial x^2} \quad \rightarrow \quad \boldsymbol{e}^2 = \frac{2}{\sqrt{3}} \hat{x}_2 \end{aligned} \tag{20a}$$

$$\boldsymbol{e}^1 \cdot \boldsymbol{e}^1 = \frac{4}{3}, \qquad \boldsymbol{e}^1 \cdot \boldsymbol{e}^2 = -\frac{2}{3}, \qquad \boldsymbol{e}^2 \cdot \boldsymbol{e}^2 = \frac{4}{3} \tag{20b}$$

Now derive using (14), (19), and (20): $d\boldsymbol{l} = \boldsymbol{e}^\alpha d\xi_\alpha = \sum_{\alpha,\beta} \boldsymbol{e}^\alpha g_{\alpha\beta} d\xi_\beta$

$$\boldsymbol{e}^\gamma = \frac{d\boldsymbol{l}}{d\xi^\gamma} \hat{x}_2 = \sum_{\alpha,\beta} \boldsymbol{e}^\alpha g_{\alpha\beta} \frac{\partial \xi^\beta}{\partial \xi^\gamma} = \sum_{\alpha,\beta} \boldsymbol{e}^\alpha g_{\alpha\gamma}$$

$$\boldsymbol{e}_1 = \boldsymbol{e}^1 g_{11} + \boldsymbol{e}^2 g_{21} \quad \rightarrow \quad \hat{x}_1 = \boldsymbol{e}^1 + \frac{1}{2} \boldsymbol{e}^2$$

$$\boldsymbol{e}_2 = \boldsymbol{e}^1 g_{13} + \boldsymbol{e}^2 g_{22} \quad \rightarrow \quad \frac{1}{2}(x_1 + x_2\sqrt{3}) = \frac{1}{2}\boldsymbol{e}^1 + \boldsymbol{e}^2$$

$$2\hat{x}_1 - \frac{1}{2}\left(x_1 + x_2\right)\sqrt{3} = \frac{3}{2}\boldsymbol{e}_1 \quad \rightarrow \quad \boldsymbol{e}^1 = \frac{4}{3}x_1 - \frac{1}{3}\left(x_1 + x_2\sqrt{3}\right) = x_1 - \frac{1}{3}x_2\sqrt{3})$$

$$\boldsymbol{e}^1 = x_1 - \frac{1}{3}x_2\sqrt{3}, \qquad \boldsymbol{e}^2 = \frac{1}{2}\left(x_1 + x_2\sqrt{3}\right) - \frac{1}{2}\left(x_1 - \frac{1}{3}x_2\sqrt{3}\right) = \frac{2}{3}x_2\sqrt{3}$$

as obtained in (20a). Alternatively

$$\boldsymbol{e}^1(\boldsymbol{\xi}) = \frac{1}{\sqrt{|g_{11}|}\left[1 - \left(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2\right)^2\right]}\left[\boldsymbol{\xi}_1 - \left(\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2\right)\boldsymbol{\xi}_2\right] = \frac{1}{1-(1/4)}\left[x_1 - \frac{1}{4}\left(x_1 + x_2\sqrt{3}\right)\right] = x_1 - \frac{1}{\sqrt{3}}x_2$$

$$\boldsymbol{e}^2(\boldsymbol{\xi}) = \frac{1}{\sqrt{|g_{22}|}\left[1 - \left(\boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_1\right)^2\right]}\left[\boldsymbol{\xi}_2 - \left(\boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_1\right)\boldsymbol{\xi}_2\right] = \frac{1}{1-(1/4)}\left[\frac{1}{2}\left(x_1 + x_2\sqrt{3}\right) - \frac{1}{2}x_1\right] = \frac{2}{\sqrt{3}}x_2$$

(21)

also in agreement with (20a). It is easily verified that $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\beta = \delta_\alpha^\beta$. The four basis vectors are illustrated in the figure 2 in which a unit circle has been drawn, and in which $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}_1$ is horizontal and $\hat{\boldsymbol{y}} = \hat{\boldsymbol{x}}_2$ is vertical. The angle between $\boldsymbol{e}^1$ and x is $-\pi/6$ rad. The angle between $\boldsymbol{e}^2$ and y is $\pi/6$ rad. As given by (19) and (20): $\boldsymbol{e}^1 \perp \boldsymbol{e}_2$ and $\boldsymbol{e}^2 \perp \boldsymbol{e}_1$.

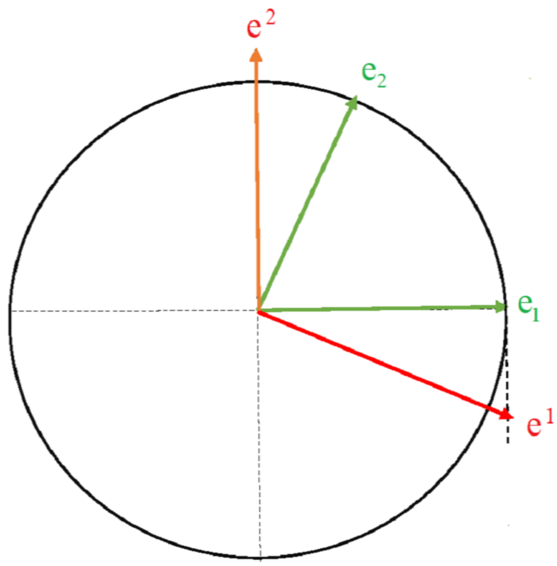

Figure 2. The four-basis vector

CONCLUSION

Basis vectors are often handier to use in an expansion of 4-vectors, among other things because they simplify the transformation to other coordinate systems. For example, the definitions (3) and (14) lead to the transformation properties between $\boldsymbol{x}$ - and $\boldsymbol{x}'$- dependent vectors and tensors

$$\boldsymbol{e}'_\beta = \frac{\partial \xi^\alpha}{\partial \xi'^\beta}\, \boldsymbol{e}_\alpha, \qquad e'^\beta = \frac{\partial \xi'^\beta}{\partial \xi^\alpha} \tag{22}$$

which in turn leads to the following for vectors:

$$\boldsymbol{V}_\alpha = \boldsymbol{e}'_\alpha \cdot \boldsymbol{V} = \frac{\partial \xi^\beta}{\partial \xi'^\alpha}\, \boldsymbol{e}_\beta \cdot \boldsymbol{V} = \frac{\partial \xi^\beta}{\partial \xi'^\alpha} V_\beta, \quad \boldsymbol{V}^\alpha = \boldsymbol{e}'^\alpha \cdot \boldsymbol{V} = \frac{\partial \xi'^\alpha}{\partial \xi^\beta}\, \boldsymbol{e}^\beta \cdot \boldsymbol{V} = \frac{\partial \xi'^\alpha}{\partial \xi^\beta} V^\beta \tag{23}$$

and the following for tensors of rank 2:

$$\boldsymbol{T} = T'^{\alpha\beta}\boldsymbol{e}'_\alpha\boldsymbol{e}'_\beta = T^{\mu\nu}\boldsymbol{e}_\mu\boldsymbol{e}_\nu \quad \rightarrow \quad T^{\mu\nu} = \frac{\partial\xi^\mu}{\partial\xi'^\alpha}\frac{\partial\xi^\nu}{\partial\xi'^\beta}T'^{\alpha\beta}$$

$$\boldsymbol{T} = T'_{\alpha\beta}e'^\alpha e'^\beta_\beta = T_{\mu\nu}\boldsymbol{e}^\mu e^\nu \quad \rightarrow \quad T_{\mu\nu} = \frac{\partial\xi'^\alpha}{\partial\xi^\mu}\frac{\partial\xi'^\beta}{\partial\xi^\nu}T'_{\alpha\beta}$$

$$\boldsymbol{T} = T'^{\alpha}_{\beta}\boldsymbol{e}'_\alpha\boldsymbol{e}'^\beta = T^{\mu}_{\nu}\boldsymbol{e}_\mu e^\nu \quad \rightarrow \quad T^{\mu}_{\nu} = \frac{\partial\xi^\mu}{\partial\xi'^\alpha}\frac{\partial\xi'^\beta}{\partial\xi^\nu}T'^{\alpha}_{\beta} \qquad (24)$$

Other tensor transformations are derived similarly from (22). As was shown in the examples, the forms (1) instead of (9a) or (12) for the contravariant ones are sometimes more convenient for nonorthogonal coordinates. The expressions for curvilinear coordinates (not necessarily orthonormal) are found relatively easily from the given expressions and their properties may be somewhat surprising.

In summary, many of the above expressions for basis vectors are not found elsewhere, and in any case not together in one coherent exposition together with some applications as given above.

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