Eigenenergies of a Relativistic Particle in an Infinite Range Linear Potential Using WKB Method

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Abstract

Energy eigenvalues for a non-relativistic particle in a linear potential well are available. In this paper we obtain the eigenenergies for a relativistic spin less particle in a similar potential using an extension of the well-known WKB method treating the potential as the time component of a four-vector potential. Since genuine bound states do not exist for positive rising potentials, our calculations are valid only for fairly low-lying levels. The transcendental eigenvalue equation that is obtained is solved using Mathematica software to get eigenenergies. Our results are compared with those for the non-relativistic case. The results may find applications besides having conceptual significance.

Keywords: Semi-classical theories and applications, Bound states, Relativistic wave equations
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Introduction

The W.K.B method that was independently proposed by G. Wentzel, H. A.Kramers and L. Brillouin is a semi classical approximation method, which is extensively used in quantum theory to obtain bound states in slowly varying potentials (Bransden and Joachain, 2004). It can only be applied to one-dimensional problems and for three-dimensional problems that are reducible to one dimension. W.K.B technique involves power series expansion of the wave function in terms of reduced Planck's constant $\hbar$. Smooth matching of the wave function is done at the points where kinetic energy of the particle becomes equal to potential energy. This leads to the quantization condition from which approximate eigenenergies could be determined for ground and excited states (Shankar, 2010). Several researchers have used this method to obtain bound states in a slowly varying potentials (Trost & Friedrich, 1997). Kagali et.al. used a new approach to calculate bound states using space integration method (Kagali et.al., 1997). The WKB treatment to relativistic wave equations is not familiar. In relativistic quantum theory, for many problems of physical interest exact solutions are not available even in one dimension. Therefore it is highly important to investigate and apply the methods of approximation, which readily works in non-relativistic cases. It is found that the extension of well known non-relativistic approximations might be possible if the relativistic wave equations are reduced to Schrödinger like form. In this article we discus relativistic approach to WKB method and we apply the same to obtain eigenenergies of a spin less particle in an infinite range linear potential.

WKB Method for Relativistic Bound States

It is of great practical importance to extend approximate methods to relativistic problems, since a very few eigenvalue problems could be solved exactly in relativistic quantum theory. In this section we extend WKB method to obtain bound states of relativistic spin zero particles. We follow the formal method of non-relativistic quantum mechanics to choose a simple well-shaped effective potential with two classical turning points x_1 and x_2 as shown in figure 1. It is shown that WKB approximation could be used in the regions for which E_{eff} is greater or less than V_{eff}, but fails at classical turning points. However connection formulae would serve at these points. In relativistic quantum theory, it is well known fact that the one-dimensional relativistic equations could be transformed into Schrödinger form as:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E_{eff} - V_{eff}\right]\psi = 0 \qquad (1)$$

Where effective energy and effective potential for a general potential $V(x)$ in vector coupling scheme could be written as,

$$E_{eff} = \frac{E^2 - m^2c^4}{2mc^2} \; ; V_{eff} = \frac{2EV(x) - V^2(x)}{2mc^2} \qquad (2)$$

Thus equation (1) could be written as

$$\frac{d^2\psi}{dx^2} + \frac{P_R^2}{\hbar^2}\psi = 0 \qquad (3)$$

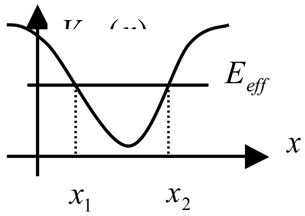

Figure 1. Schematic representation of well shaped effective potential with two classical turning points x_1 and x_2

When effective potential varies smoothly, the wave function may be written as,

$$\psi(x) = \varphi(x)\, e^{\left[\pm\frac{i}{\hbar}\int P_R(x)\,dx\right]} \qquad (4)$$

Where $\phi(x)$ is slowly varying function. This is the basis of WKB method. Carrying out the details, we obtain the quantization condition as

$$\int_{x_1}^{x_2} P_R(x)\,dx = \left(n + \frac{1}{2}\right)\pi\hbar \qquad (5)$$

Application of WKB Method to Obtain Bound States of A Particle In An Infinite Range Linear Potential.

For a relativistic spin zero particle in an infinite range linear potential defined by

$$V(x) = k|x| \tag{6}$$

The schematic plot of effective potential with effective energy would be as shown in figure 2.

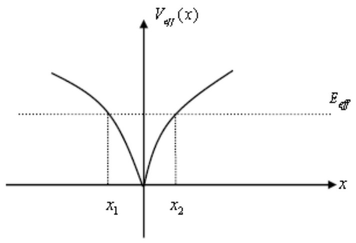

Figure 2: Schematic representation of effective potential with effective energy for an infinite range linear potential.

The momentum of a relativistic particle with a general potential $V(x)$ in vector coupling scheme could be written as,

$$P_R = \sqrt{\left[\frac{(E - V(x))^2 - m^2c^4}{c^2}\right]} \tag{7}$$

At turning points $E_{eff} - V_{eff} = 0$, taking $x = \frac{E}{k}y$, we obtain

$$x_{1,2} = \mp\left(1 - \frac{mc^2}{E}\right) \tag{8}$$

Thus by using quantization rule, we get

$$\frac{2E}{kc}\int_0^{1-\frac{mc^2}{E}} \left[E^2(1-y)^2 - m^2c^4\right]^{1/2} dy = \left(n + \tfrac{1}{2}\right)\pi\hbar \tag{9}$$

On simplifying the above integral we get,

$$\begin{aligned} &E\sqrt{E^2 - m^2c^4} - \left(n + \tfrac{1}{2}\right)\pi\hbar kc \\ &\quad = m^2c^4\left[Log\left(E + \sqrt{E^2 - m^2c^4}\right) - Log\left(mc^2\right)\right] \end{aligned} \tag{10}$$

The above transcendental equation is solved graphically using Mathematica (Wolfram, 1996). To obtain numerical solution we take $m = c = \hbar = k = 1$. The graphical solution for eigenenergies is obtained by plotting LHS of equation (10) versus its RHS. It is as shown in figure: 3.

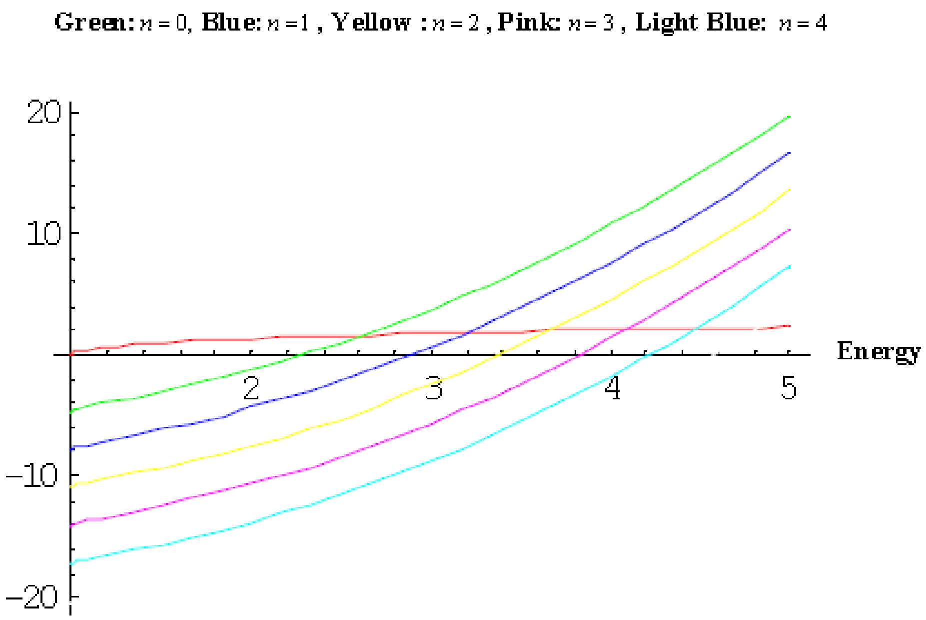

Figure 3. Graphical solutions to eigenenergy of a relativistic spin zero particle in an infinite range linear potential.

In the following table, the eigenenergies of non-relativistic (Shankar, 2010), relativistic particle evaluated by WKB method are compared with that of exact value (Casaubon, 2007) for n = 1 and n = 3.

n	E_{WKB} non-relativistic	E_{WKB} relativistic	E exact
1	1.842	1.893	1.85575
3	3.240	2.827	3.24457

Results and Discussions

We have extended WKB method to relativistic situation (Klein Gordon approach) and we have applied the method to obtain eigenenergies of a relativistic spin zero particle in an infinite range linear potential well. Also we have compared these eigenenergies with that of exact values. We found that the eigenenergies obtained are well suited for relatively lower values of n, but with increase in n, the difference between eigenenergies estimated from WKB and exact value increases due to variation in the effective potential. Further one should be careful that very high value of n for the potential of this kind may lead to a scattering situation instead of yielding bound states. Further the relativistic version of WKB method surely serves as an important technique to obtain eigenenergies of particles in specific potentials for which solutions are not available in relativistic regime. It would be interesting to apply the method for relativistic spin half particles.

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