A New Framework to Study The Wave Motion Of Flexible Strings in the Undergraduate Classroom Using Linear Elastic Theory

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Abstract

When deriving the equation describing the transverse motion of a one-dimensional vibrating elastic string, introductory physics textbooks often assume constant tension and/or small amplitude vibrations. However, these simplifying assumptions are not only unnecessary, but they overlook the elastic nature of the tension and yield an inconsistent derivation of the potential energy density. Because of these assumptions, the derivation of the wave equation and the potential energy density use two different levels of mathematical approximation. In addition, students often get confused as of how a string can carry elastic potential energy if the underlying assumption is that the tension is constant. In this work, we present a mathematically consistent derivation of the wave equation and potential energy density for the vibrating string. Our approach is adequate for physics and engineering introductory courses. We emphasize throughout the derivations the role of elasticity and we propose a simple experiment where students can use wave theory to predict the elastic properties of strings. We also use our framework to illustrate under which conditions longitudinal waves can be neglected for strings that obey Hooke's law of elasticity. We show that a small transverse amplitude vibration does not immediately justify neglecting longitudinal motion.

Keywords: Wave equation, waves on a string, transverse waves

INTRODUCTION: WAVES ON A STRING

In 1959, J.B. Keller (Keller, 1959) noticed that strings made of a "perfectly elastic" material can undergo pure transverse vibrations. For a perfectly elastic material, the tension on the string is directly proportional to the stretch of the string, such that at zero tension the string has zero length[1]. Therefore, having a string made of a perfectly elastic material is a sufficient condition for transverse motion that does not require further assumptions (Keller, 1959; Antman, 1980; Luke, 1992). But over the years, it seems that the power of this discovery has been overlooked by the physics education community. Using Newton's Laws and some basic knowledge of partial derivatives popular lower division undergraduate textbooks guide students through the derivation of the linear wave equation that describes the motion of a vibrating string. But those undergraduate derivations often invoke unnecessary assumptions such as constant tension in the string or a small vibrations approximation (Walker, Halliday, & Resnick, 2018; Giancoli, 2020; Katz; Knight, 2016; Serway & Jewett, 2019; Tipler & Mosca, 2007; Alonso & Finn, 1983; Wolfson & Richard, 2011; French, 2003; Shankar, 2014)[2].

We are not the first ones to have taken notice of the problems with the current undergraduate derivations and alternatives have been suggested (Clelland & Vassiliou, 2013; Yong, 2006). But although correct approaches to the derivation of the linear wave equation in an elastic string can be found in the literature, these explanations are often beyond the mathematical knowledge possessed by students taking introductory calculus-based physics courses. In addition, students in introductory courses are often asked to perform experiments of transverse standing waves on a string[3] where it is evident that the waves undergo large amplitude displacements, yet students are instructed to study such experiments using the 'linear small-amplitude theory' that they have seen in their textbooks. Therefore, we see the need to have a simple, yet rigorous derivation at the introductory level that clearly identifies the assumptions of pure transverse motion while simultaneously connecting the theory to experiments that can be done in the undergraduate lab.

A second point that we want to address in this paper is the inadequacy of the current undergraduate derivations of the potential energy density. A common misconception that students have regarding the potential energy of transverse waves in an elastic string is: '*just like a simple harmonic oscillator, maximum potential energy of a wave occurs when the kinetic energy is a minimum*'. As addressed in (Ng, 2010), this incorrect assertion comes from trying to obtain the potential energy density from a model that treats every string element as an 'independent' oscillating mass. The 'independent' oscillating mass model is widely used because it reproduces the correct result for the average total energy per unit wavelength, but at the expense of erroneously leading students to think that the energy density on the string is a constant. The 'independent' mass oscillating model cannot explain how energy can propagate in a travelling

[1]Perfectly elastic materials are not the same as linearly elastic. See section (5).

[2]These sorts of assumptions can also be found in physics and math textbooks that are often used at upper division undergraduate and graduate level. See for example (Garret, 2020; Myint U. & Debnath, 2007; Lee & W., 1988).

[3] The experiments often use a slinky or soft elastic strings.

wave (Ng, 2010). Unfortunately, this incorrect model is still presented in the literature (Giancoli, 2020; Katz; Ryan, Neary, RInaldo, & Olivia, 2019; Ling, Sanny, & Moebs, 2016; hyperphysics, 2022). Other popular textbooks either skip the topic of potential energy density altogether (Knight, 2016; Tipler & Mosca, 2007; Young & Freedman, 2016) or omit the derivation but show the final result (Walker, Halliday, & Resnick, 2018; Serway & Jewett, 2019). This is unfortunate, since as educators we should avoid the handwaving expression "it can be shown", and derivations should always be presented when possible. Finally, there is a set of undergraduate sources that derive the potential energy based on a correct physical approach of the string's stretching, but regularly those derivations assume a constant tension along the string and a second-order expression for the change in length of the string (French, 2003; Shankar, 2014; Fowler; Salsa, 2008; Caamano Withall & Krysl, 2016). These assumptions are not only unnecessary but quite confusing. First, the constant tension assumption is contradictory as any change in length of a string will result in some change in tension. Second, the derivations are mathematically inconsistent. The second-order expression for the string's length is only used when deriving the potential energy density, but the change in length is neglected in the rest of the model while deriving the linear wave equation. These two remarks may be familiar to instructors and the astute physics students. In fact, students who are critical and deep thinkers can be found among those who find such derivations unsatisfying. In this paper we present an alternative derivation of the potential energy density that uses the definition of the elastic potential energy (generalized Hooke's Law). Our approach is simple, straightforward, and suitable for an introductory physics course.

We have organized our paper as follows. In section (2) we define the general features of a model for waves on an elastic string. We introduce the concepts and assumptions that are shared by the current undergraduate derivations and our new proposed approach. In section (3) we reproduce an example of current undergraduate derivations for the wave equation and the potential energy density assuming small vibrations and constant tension. We highlight the points that can lead to student's confusion and misunderstanding. In section (4) we present our new mathematically consistent approach that is meant to remedy the issues we highlight in section (3). We consider strings made of linear elastic materials, and clearly identify under which conditions waves can be approximated as purely transverse. In section (5) we propose a low-cost experiment where students will quantitatively explore the elastic properties of the string.

GENERAL MODEL FOR TRANSVERSE WAVES ON A STRING

The @slinky, ropes, wires, yarn, cables, and rubber bands are commonly found in physics classroom demonstrations and laboratory activities. If certain conditions are met, these materials, and many others, can be effectively described using a one-dimensional string model (O'Reilly, 2017). In this section, we focus on the features of the string model that are relevant to the study of transverse vibrations in an introductory calculus-based physics course. This description is shared by current undergraduate derivations (section 3) and our new proposed approach (section 4).

We think of the string as a continuum and homogeneous object where its length is much larger than its cross-sectional area (one-dimensional object). The strings mass distribution can be characterized mathematically by a uniform linear mass density function μ_o, where the integral of μ_o over the length of a string's segment gives the mass of the segment[4]. The string is assumed to be elastic[5], and conservation of angular momentum implies the tension is tangential to the string. The string has negligible area moments of inertia such that the string carries no resistance to bending and torsion[6]. Hence, the only mode of deformation for the elastic string is the stretch caused by tension. Finally, the weight of the string, friction, air drag, and any other dissipative forces are neglected in the model.

Now, let us consider a string tautly stretched between two posts (Figure 1). The ends of the string are fixed to the posts. We construct a *x-y* coordinate system and set the origin at the left post ($x = 0, y = 0$) and lay out the x coordinate along the line directly connecting the two posts, so that the post on the right is located at the position ($x = L_o, y = 0$). Figure 1a displays the string in the equilibrium straight horizontal configuration where no waves are present. In the equilibrium configuration the string is under tension T_o and stretched to a length L_o that is greater than its natural unstretched length $\hat{L}$. Figure 1(a) also shows in red a small element of mass δm and length δx. The string is set to vibrate by introducing a disturbance such as plucking it. Figure 1b shows a sample wave snapshot on the deformed configuration where the string is stretched to a length $L > L_o$. Figure 1(b) also shows the red element of mass δm, that has been stretched to a length δs. In Figure 2 we zoom into the mass element δm in the string's deformed configuration. Without any loss of generality, we define the coordinates a point A to be $[x, y(x)]$ and the direction of the tension at point A to be determined by the angle $\theta_A = \theta(x)$. Then point B will have coordinates $[x + \delta x, y(x + \delta x)]$, and the direction of the tension will be characterized by the angle $\theta_B = \theta(x + \delta x)$:

We apply Newton's law of motion in the horizontal and vertical directions:

$$T(x)\cos\theta_B - T(x)\cos\theta_A = \delta m a_x, \qquad \text{Eq [1]}$$

$$T(x+\delta x)\sin\theta_B - T(x + \delta x)\sin\theta_A = \delta m a_y, \qquad \text{Eq [2]}$$

replacing $\delta m = \mu_o \delta x$, $\theta_A = \theta(x)$, and $\theta_B = \theta(x + \delta x)$:

$$\frac{T(x+\delta x)\cos\theta(x+\delta x) - T(x)\cos\theta(x)}{\delta x} = \mu_o a_x, \qquad \text{Eq [3]}$$

[4] Due to local conservation of mass along the string, as the string deforms the linear mass density will change.
[5] Different constitutive models can be used to describe the elastic nature of a string: perfectly elastic, linear elastic, hyper-elastic, etc (O'Reilly, 2017). In the current paper we do not discuss the dynamics of inextensible strings.
[6] This detail can be omitted when presenting it to an introductory physics audience without background on strength of materials.

$$\frac{T(x+\delta x)\sin\theta(x+\delta x) - T(x)\sin\theta(x)}{\delta x} = \mu_o a_y. \qquad \text{Eq [4]}$$

In the limit as the length of the string element vanishes $\delta x \to 0$, the left-hand side of the above equations reduce to a spatial derivative:

$$\frac{\partial(T(x)\cos\theta(x))}{\partial x} = \mu_o a_x, \qquad \text{Eq [5]}$$

$$\frac{\partial(T(x)\sin\theta(x))}{\partial x} = \mu_o a_y. \qquad \text{Eq [6]}$$

Up to this point we have not invoked any assumption regarding the behavior of the string, the two expressions above are true for the dynamics of *any one-dimensional string.* The above equations describe the motion in the horizontal and vertical direction, but they require knowing the physical law describing $T(x)$.

Geometric Description of Pure Transverse Motions

Consider the string in Figure 1. If all points along the string undergo motion perpendicular to axis of the string (horizontal axis), then the wave is said to be purely transverse. Consider a small element of mass δm undergoing pure transverse motions as shown in Figure (2). The total stretch of this mass element is given by $\delta s - \delta x$, where the length of the distorted element is given by:

$$\delta s = \sqrt{\delta x^2 + \delta y^2} = \delta x\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}. \qquad \text{Eq [7]}$$

In the limit as $\delta x \to 0$, for an infinitesimally small element of initial undeformed length dx, the final deformed length would be given by $ds = \sqrt{dx^2 + dy^2} = \sec\theta(x)\, dx$.

We conclude this section by stating the two conditions for pure transverse motion on a string:

(1) The geometric requirement:

$$dx = \cos\theta(x)\, ds. \qquad \text{Eq [8]}$$

(2) From the dynamics equation of motion Eq (5), the horizontal component of the acceleration is zero, if and only if the horizontal component of the tension ($T_x = T(x)\cos\theta(x)$) is a constant independent position.

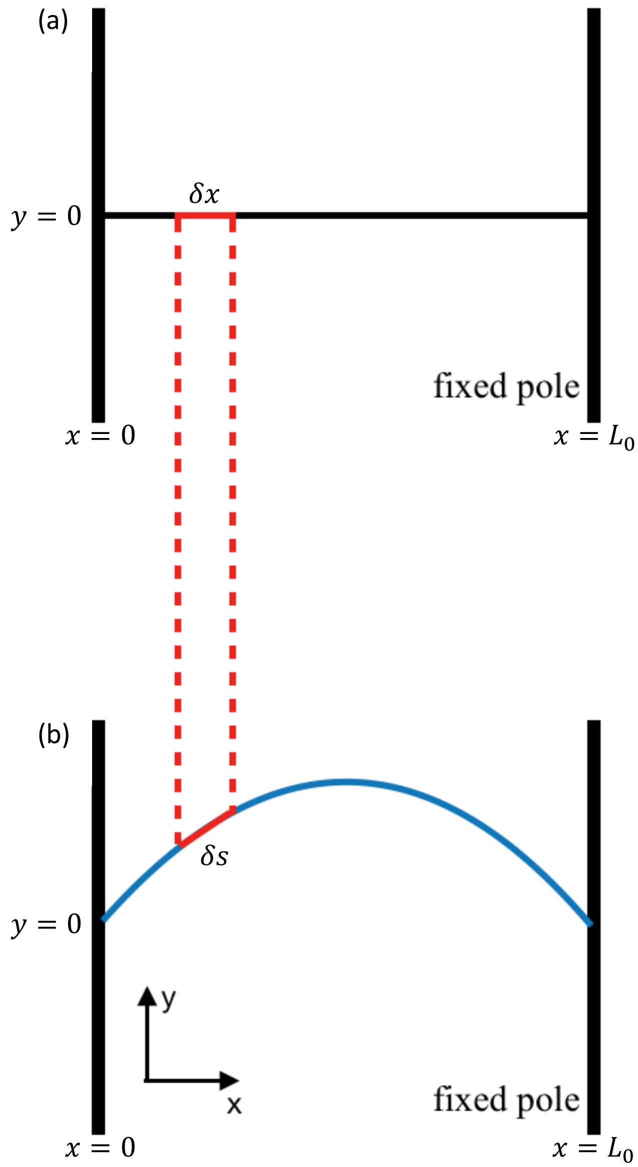

Figure 1. *A flexible string of natural unstretched length $\hat{L}$ is fixed in space to poles. Part (a) shows the string in the static equilibrium configuration without any waves. The string is under a tension T_0, and it has been stretched to an equilibrium length L_0 larger than $\hat{L}$. A small element of mass δm and length δx from the string is colored in red. Part (b) shows a snapshot of a wave present in the string's deformed configuration. The wave disturbance leads to further local stretching as depicted by the new extended length δs of the mass element δm. The string's total length in the deformed state is L.*

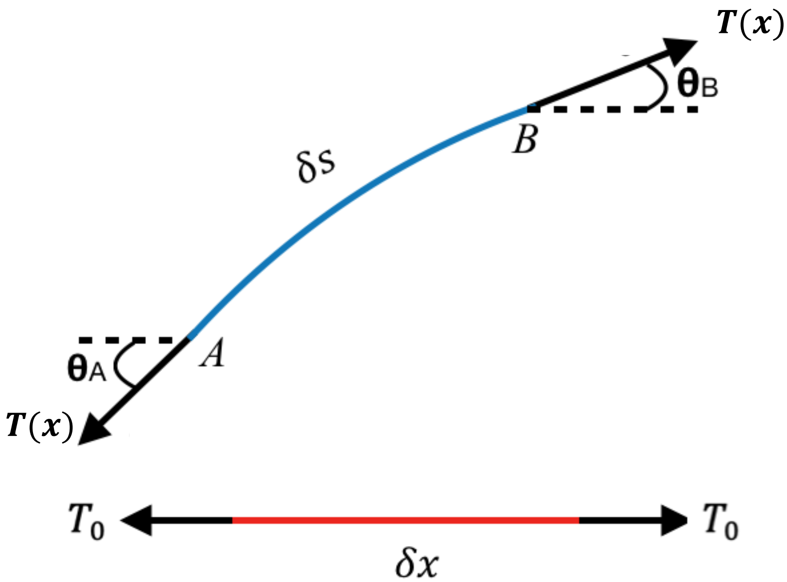

Figure 2. *A small mass element δm is shown in the static equilibrium (red) configuration and deformed (blue) configuration. As a wave travels through the string, the element δm is deformed and it's the length changes from δx to δs.*

CONFUSING UNDERGRADUATE DERIVATIONS

Next, we will follow closely the approach taken in the calculus-based physics textbook (Shankar, 2014). This approach is representative of what is presented in many introductory textbooks when deriving the wave equation on a linear elastic string[7] and the potential energy density of the transverse wave[8]. We use the approximation small amplitude string vibration to justify the following conditions:

(a) The string deflections about the equilibrium position are assumed to produce negligible effects on the tension, such that the tension remains approximately constant and equal to the equilibrium value $T(x) = T_0$.

(b) The small deflections of the string are characterized by small slopes along the string $\mathrm{y}'(\mathrm{x}) = \tan(\theta) \approx \theta$, and therefore in the following derivations we will retain terms up to linear order on the slopes such that[9], $\sin(\theta) \approx \theta \approx \tan(\theta) = y'(x)$ and $\cos(\theta) \approx 1$.

From (a) and (b) the equations of motion Eq [5] and Eq [6] reduce to:

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[7] See for instance (Walker, Halliday, & Resnick, 2018; Giancoli, 2020; Katz; Knight, 2016; Serway & Jewett, 2019; Tipler & Mosca, 2007; Alonso & Finn, 1983; Wolfson & Richard, 2011; French, 2003; Ling, Sanny, & Moebs, 2016)

[8] See for instance (French, 2003; Fowler; Salsa, 2008; Caamano Withall & Krysl, 2016)

[9] We use the (‘) accent notation to represent partial derivative with respect to position (x).

$$\frac{d(T_0)}{dx} = \mu_o a_x = 0. \qquad \text{Eq [9]}$$

$$T_0 \frac{d(y'(x))}{dx} = T_0 y''(x,t) = \mu_o a_y. \qquad \text{Eq [10]}$$

As evident from equation, the horizontal component of the force is constant and equal to T_o, and therefore there is no motion in the horizontal direction ($a_{x,} = 0$). It is important to emphasize that in the typical undergraduate derivation, it is this level of approximation ($\cos\theta \approx 1$), that justifies pure transverse displacements.

We rewrite the acceleration in the vertical direction $a_y = \ddot{y}(x,t)$ as the second derivative of the position coordinate $y(x,t)$ with respect to the time variable[10], such that equation reduced to the famous linear wave equation in a flexible string:

$$y''(x,t) = \frac{\mu_0}{T_0}\ddot{y}(x,t). \qquad \text{Eq [11]}$$

This concludes the 1st part of the derivation.

Next, we present the derivation of the potential energy density stored in a string when waves are present. We follow the steps of the derivation as it typically done by the undergraduate physics education community.

As a wave travels through the string, the elements of the string are assumed to have a small but quantifiable stretch. Therefore, the potential energy stored in the string is elastic in nature and it equal to the work done by the tension during the stretch. We consider a small element of mass δm as shown in Figure (1), where the points along the string are assumed to move only along the transverse direction. The extension of the string is given by Eq. (7), and for small vertical deflections $\frac{\delta y}{\delta x} \ll 1$, then the length distorted length of the element would be:

$$\delta s \approx \delta x\left(1 + \frac{1}{2}\left(\frac{\delta y}{\delta x}\right)^2\right). \qquad \text{Eq [12]}$$

Since the tension T_0 is assumed to remain constant, then the work done by the tension during the stretch of the element of mass δm from δx to δs is:

$$\delta W = T_o(\delta s - \delta x) = T_o \delta x\left(\frac{1}{2}\left(\frac{\delta y}{\delta x}\right)^2\right). \qquad \text{Eq [13]}$$

Defining the potential energy density (u_p) as the work per unit length, and taking the limit as $\delta x \to 0$ we obtain the classical result:

[10]We use the dot accent notation to represent partial derivative with respect to time.

$$u_p = \frac{1}{2} T_0 (y'(x)^2). \qquad \text{Eq [14]}$$

This concludes the derivation of both the wave equation and the potential energy density.

Next, we go over the points in the derivation that can leave students confused and unsatisfied. First, students with background in properties of materials will find it contradictory to have a constant tension while simultaneously allowing the string to stretch. Even for small deformations the linearized theory of elasticity tells us that the stress (tension) depends on the strain (stretch). In addition, students might look back at the derivation and find a mathematically inconsistent use of the small angle approximation. In section 2.1 we found that for pure transverse motion the string elements must satisfy $dx = \cos\theta(x)\, ds$. But when deriving the wave equation Eq [1], we have used the $\cos\theta(x) \approx 1$ such that the longitudinal displacements of the string are neglected. If we were to consistently use the same level of approximation (when deriving the potential energy) it would imply that the string has negligible stretch, and the potential energy density should in fact be zero. It is clear then, that to have a quantifiable stretch one needs to use a higher order expansion of $cos\,\theta \approx 1 - \frac{\theta^2}{2}$, such that

$$ds = \frac{dx}{\cos(\theta)} \approx \left(1 + \frac{\theta^2}{2}\right) dx = \left(1 + \frac{1}{2} y'(x)\right) dx. \qquad \text{Eq [15]}$$

Some questions that might pop in a student's mind are: what kind of material satisfy the assumption of constant tension? why do we to use two different approximations for the $\cos\theta(x)$ when we are talking about the same physical system? Is the assumption of constant tension T_o even compatible with the assumption that the string undergoes pure transverse displacements ($\cos\theta(x) \approx 1$)?

NEW APPROACH: PURE TRANSVERSE MOTION AND PERFECTLY ELASTIC MATERIALS

The Linear Wave Equation

In this section, we consider an elastic string that in addition to the general description given in section (2), it also obeys the constitutive law of a linear elastic material (Hooke's law).

We divide the string into N elements of equal mass δm, where each element 'i' behaves as linear elastic spring. In the absence of any applied tension the natural unstretched total length of the string is $\hat{L}$, and $\delta\hat{x}$ is the natural unstretched length of the small mass element δm.

Then as the element is stretched by a distance Δ, the tension T on the element is given by Hooke's constitutive law $T = k(\Delta)$, where k is the constant of elasticity[11] that depends on the material properties and is specified empirically.

Fig 1a shows the static equilibrium configuration of the string where the length total length is L_o and each mass element has been uniformly stretched by a distance $\Delta = \delta x - \delta\hat{x}$. Therefore, the equilibrium uniform tension T_o is given by:

$$T_o = k\left(L_o - \hat{L}\right) = k(\delta x - \delta\hat{x}). \qquad \text{Eq [16]}$$

Fig 1b shows the deformed stretched configuration of the string where the total length is $L > L_o$, and the mass element have been stretched by a distance $\Delta = \delta s - \delta\hat{x}$. Therefore, the tension in the deformed configuration is:

$$T(x) = k(\delta s - \delta\hat{x}). \qquad \text{Eq [17]}$$

Solving for the k in terms of the equilibrium tension T_o we find:

$$T(x) = T_o\frac{(\delta s - \delta\hat{x})}{(\delta x - \delta\hat{x})}. \qquad \text{Eq [18]}$$

After some algebra (see Appendix A2), and taking the limit as $\delta x \to 0$, the tension for an infinitesimal element of the string in the deformed configuration can be written as:

$$T(x) = T_o\left[\frac{ds}{dx} + \left(\frac{\hat{L}}{L_o - \hat{L}}\right)\left(\frac{ds}{dx} - 1\right)\right]. \qquad \text{Eq [19]}$$

Next, we check if the above expression simultaneously satisfies the two conditions for pure transverse motions given in section 2.1. We impose the geometric condition for a transverse wave $ds = \cos\,\theta(x)dx$, such that the horizontal and vertical components of the tension are:

$$T_x = T(x)\cos\theta = T_o\left[1 + \left(\frac{\hat{L}}{L_o - \hat{L}}\right)(1 - \cos\theta)\right], \qquad \text{Eq [20]}$$

$$T_y = T(x)\sin\theta = T_o\left[\tan\theta + \left(\frac{\hat{L}}{L_o - \hat{L}}\right)(\tan\theta - \sin\theta)\right], \qquad \text{Eq [21]}$$

were $\theta = \theta(x)$ is a function of the x-coordinate. We see that T_x is spatially dependent and it will yield longitudinal accelerations. Assuming small amplitude vibrations does not warrant an approximately constant horizontal tension. Even for $\cos\theta \approx 1$, if the string static equilibrium stretch L_o and the natural unstretched length $\hat{L}$ of the string are approximately equal ($L_o \sim \hat{L}$),

[11] For a string with cross sectional area $\hat{A}$, the constant $k = \frac{E\hat{A}}{\delta\hat{x}}$, where E is Young's modulus (E).

the second term in the expression T_x can be order of unity and the spatially variations of T_x can be significant.

So, under which conditions is the constant horizontal tension $T_x \approx T_o$ approximation justified? If the equilibrium stretched length of the string is much larger than the natural length $\left(L_o \gg \hat{L}\right)$, then second term in the expression for T_x vanishes in comparison to unity [12]:

$$\frac{1}{\varepsilon_o} = \left(\frac{\hat{L}}{L_o - \hat{L}}\right) \approx 0, \qquad \text{Eq [22]}$$

where ε_o is known as the engineering strain in the equilibrium configuration and it measures the extension in going from the natural unstretched configuration $\hat{L}$, to the equilibrium stretched configuration of the string L_o. For $L_o \gg \hat{L}$, the tension $T(x)$ in the deformed configuration is given:

$$T(x) \approx T_o \frac{ds}{dx}. \qquad \text{Eq [23]}$$

Strings that obey the above constitutive law for the stretching behavior are known as *perfectly elastic*. These materials can be modelled as having a negligible length in the absence of tension $(\hat{L} \approx 0)$[13]. Examples of such behavior can be seen in soft pre-tensioned springs such as the @slinky. Finally, for a *perfectly elastic string* the vertical component of the tension reduces to:

$$T_y = T(x) \sin\theta(x) = T_0 \tan\theta(x) = T_0\, y'(x). \qquad \text{Eq [24]}$$

Using $a_y = \ddot{y}(x,t)$, and plugging T_y into the equation of motion Eq [6], we recover the one-dimensional linear wave equation:

$$y''(x,t) = \frac{\mu_0}{T_0}\ddot{y}(x,t), \qquad \text{Eq [25]}$$

Therefore, for special case of elastic strings where the initial stretch is large compared with the natural length ($\varepsilon_o \gg 1$), the linear wave equation describes the vibrating motion of a transverse wave exactly. This is the case even for moderate amplitude vibrations when $\cos\theta \ll 1$.

Potential Energy Density Using The Expressions For Springs (Non-Calculus)

Just like before, we continue to model the string as a linear elastic material that has been divided into N small mass elements δm. Each element 'i' behaves like Hookean spring. A change in length Δ in the mass element will result in a change in elastic potential energy of $k(\Delta)^2/2$, were k is the elastic constant for the material. In the absence of any loads an element is initially at its natural

[12] Note that since the string is under a uniform stretch in the equilibrium configuration the requirement $L_o \gg \hat{L}$ is the same as $dx \gg d\hat{x}$.

[13] Note that as $\delta\hat{x}$ approach zero, k will remain finite if $E\hat{A}$ approach zero as well. This means that the zero length materials are good approximations for soft elastic strings with negligible cross-sectional area $\hat{A}$ and low Youngs modulus E.

length $\delta\hat{x}$. Then a constant tension T_0 is applied to bring the string to the equilibrium stretched state as shown in Fig 2. Under tension T_0, the element stretches from $\delta\hat{x}$ to a length δx, and its potential energy in the equilibrium stretched configuration is:

$$\delta U_{oi} = \frac{1}{2}k(\delta x - \delta\hat{x})^2. \qquad \text{Eq [26]}$$

During the wave motion, the tension is increased further, stretching the string element to a length δs and its potential energy in the deformed state is:

$$\delta U_{si} = \frac{1}{2}k(\delta s - \delta\hat{x})^2. \qquad \text{Eq [27]}$$

The potential energy 'stored' in the elements of the string due to the wave traveling (δU_i) is given by the difference in the energy before and after deformation:

$$\delta U_i = \delta U_{si} - \delta U_{oi} = \frac{1}{2}k(\delta s - \delta\hat{x})^2 - \frac{1}{2}k(\delta x - \delta\hat{x})^2. \qquad \text{Eq [28]}$$

Using the definition for the deformed tension $T(x)$ and the equilibrium tension T_o we can rewrite the expression above as:

$$\delta U_i = \frac{1}{2}T(x)(\delta s - \delta\hat{x}) - \frac{1}{2}T_o(\delta x - \delta\hat{x}). \qquad \text{Eq [29]}$$

For the special case of *perfectly elastic string* the natural length $\delta\hat{x}$ is negligible in comparison to the stretched length δx and δs and the stretched tension is given by $T(x) = T_o\frac{\delta s}{\delta x}$. Therefore, the expression for δU_i reduces to:

$$\delta U_i = \frac{1}{2}T_o\frac{\delta s}{\delta x}(\delta s) - \frac{1}{2}T_o(\delta x) = \frac{1}{2}T_o\left[\left(\frac{\delta s}{\delta x}\right)^2 - 1\right]\delta x, \qquad \text{Eq [30]}$$

since $\delta s^2 = \delta y^2 + \delta x^2$, then

$$\delta U_i = \frac{1}{2}T_o\left(\frac{\delta y}{\delta x}\right)^2\delta x. \qquad \text{Eq [31]}$$

In the limit as δx becomes an infinitesimal element along the string we get the potential energy per unit length (density) to be:

$$\frac{dU}{dx} = \frac{1}{2}T_o\big(y'(x)\big)^2. \qquad \text{Eq [32]}$$

Potential Energy Density Using The Concept Of Work (Calculus)

Some students might not wonder why not derive the potential energy density using tension. Ultimately it is the work done by tension against the stretching which gives rise to the elastic energy. Below we show an alternative derivation that might satisfy the curiosity of those students.

As done earlier our starting point is a string divided into N small mass elements δm. Each element 'i' undergoes pure transverse displacements and the deformation can be characterized by the deformation angle θ_i (See Fig 3).

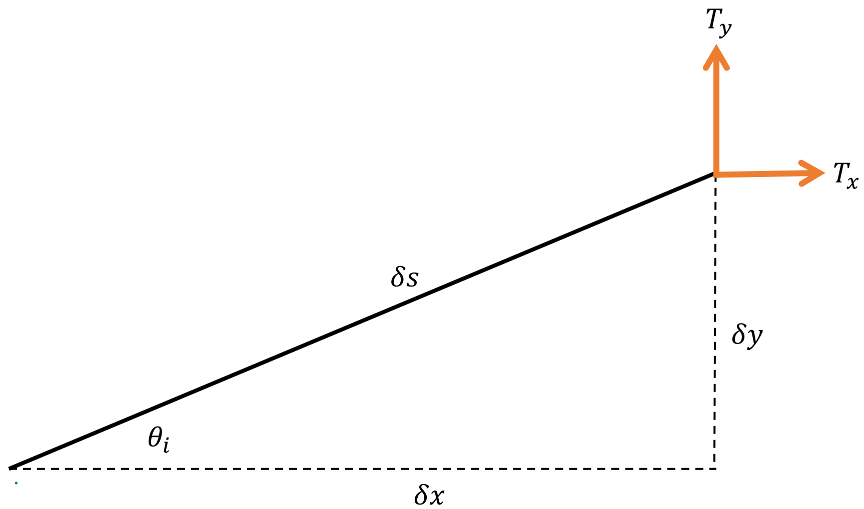

Figure 3. *The i^th element of a string that undergoes pure transverse displacement. The string has been divided into equal N equal elements of mass δm.*

Due to the pure transverse motion argument, the infinitesimal displacement vector has only a component along the vertical direction $\overrightarrow{dr} = dy\,\hat{\jmath}$, and therefore the potential energy of element 'i' that is displaced from equilibrium by a pure transverse displacement δy is given by:

$$\delta U_i = \int \overrightarrow{dT} \cdot \overrightarrow{dr} = \int_0^{\delta y} T_y\, dy = \int_0^{\delta y} T_o \tan(\theta_i)\, dy. \qquad \text{Eq [33]}$$

To evaluate the above integral, we notice that as an element 'i' deforms, its horizontal projection length δx remains constant while the deformation angle θ_i increase is related to the vertical coordinate y by the following relation:

$$\tan\big(\theta_i(y)\big) = \frac{y}{\delta x}. \qquad \text{Eq [34]}$$

Therefore, the potential energy of the 'i^th' element reduces to:

$$\delta U_i = \frac{T_o}{\delta x}\int_0^{\delta y} y\, dy = \frac{1}{2} T_o \left(\frac{\delta y}{\delta x}\right)^2 \delta x. \qquad \text{Eq [35]}$$

Which is the same expression we obtained in section 4.1.

FROM THEORY TO EXPERIMENT: LINEAR ELASTIC MATERIALS

In this section, we propose an experiment using linear elastic strings. Linear elastic theory (Hooke's law) describes the behavior of materials that return to their original shape after the external deforming force is removed. We have used a set of tensional springs with small-cross sectional area as the 1D 'string'. These springs are commonly used by students in labs when exploring Hooke's Law[14]. Our proposed laboratory activity will further emphasize the stretchy nature of the string, and it will help students connect wave theory with elasticity. During wave motion elements of the string stretched by an amount Δ, the tension along that portion of the string is given by Hooke's law $T = k\Delta$, where k is the 'stiffness' constant. The goal of the experiment is to characterize the constant k. Before starting the activity, students should also be familiar with standing waves on a string. For this experiment students will need:

5.1 Equipment:

- Elastic String (we used a tensional string: https://www.pasco.com/products/lab-apparatus/mechanics/springs-and-oscillations/se-8749)
- Wave Driver (we used the mechanical oscillator: https://www.pasco.com/products/lab-apparatus/waves-and-sound/ripple-tank-and-standing-waves/sf-9324)
- Wave generator (we used https://www.pasco.com/products/lab-apparatus/waves-and-sound/ripple-tank-and-standing-waves/wa-9867)
- A fixed support.

The string (tensional spring) is attached at one end to a mechanical oscillator and the other end attached to a fixed support. The ends of the string are at the same height such that the string is hanging in a horizontal position. L_0 is the distance between the supports. As L_0 is increased the tension on the string will increase. A picture of the experimental set up is shown in Fig 4.

[14] We note that the proposed wave experiments can be replicated using strings made out of rubber bands, elastic cords, and other fabrics. To a good level of approximation, most materials in our macroscopic world can be modelled as linear elastic when the deformations are small.

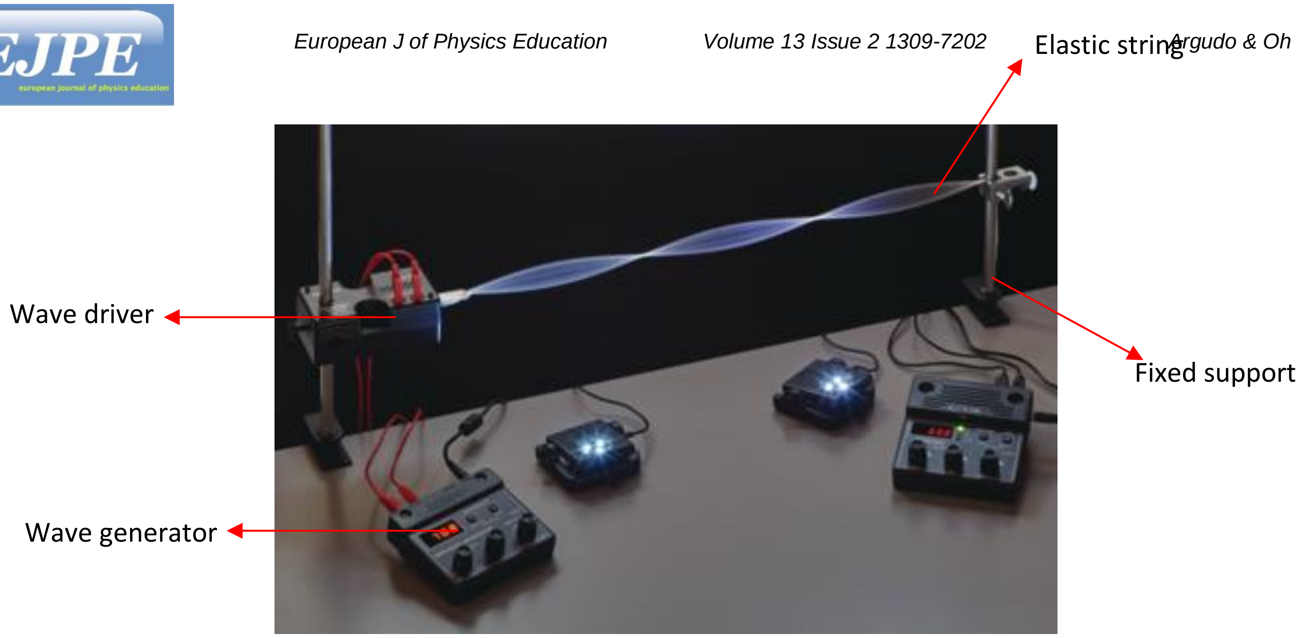

Figure 4: *Suggested experimental set up. Image taken from https://www.pasco.com/products/lab-apparatus/waves-and-sound/ripple-tank-and-standing-waves/wa-9867*

Procedure

The procedure for the experiment is simple. Students are instructed to find the 1st harmonic wave on the string and document the fundamental frequency f_1 as the distance between the fixed supports L_0 is modified. Then k can be found by quantifying the relationship between f_1 and L_0. The fundamental frequency (1st harmonic) of the standing wave for a linear elastic string is:

$$f_1 = \frac{1}{2L_0}\sqrt{\frac{T_0}{\mu_0}}. \qquad \text{Eq [36]}$$

For a uniform density string $\mu_0 = m/L_0$ and for a linear elastic string the tension is $T_0 = k(L_0 - \hat{L})$, the expression above can be rewritten in the form $(f_1)^2 = kq$, were $q = \frac{1}{4m}\left(1 - \frac{\hat{L}}{L_0}\right)$ is the independent variable that depends directly on the initial stretch of the string. Then a plot of $(f_1)^2\ vs.\ q$ will yield the elastic constant as the slope of the graph. A sample scenario of an experiment is shown in Table 1. The comparison of the experimental data from Table 1 and the theoretical prediction is plotted in Figure 5.

Table 1: *Fundamental frequencies f_1 corresponding to the standing waves of an elastic string as a function of the initial stretch L_o. The string has mass m = 2.5 [g], and undeformed length $\hat{L} = 5.5$ [cm] and a stiffness constant k = 5 [N/m]*

Frequency f_1 [Hz]	16.0	18.0	19.5	20.1	20.5
Initial Stretch L_o	$2\hat{L}$	$3\hat{L}$	$4\hat{L}$	$5\hat{L}$	$6\hat{L}$

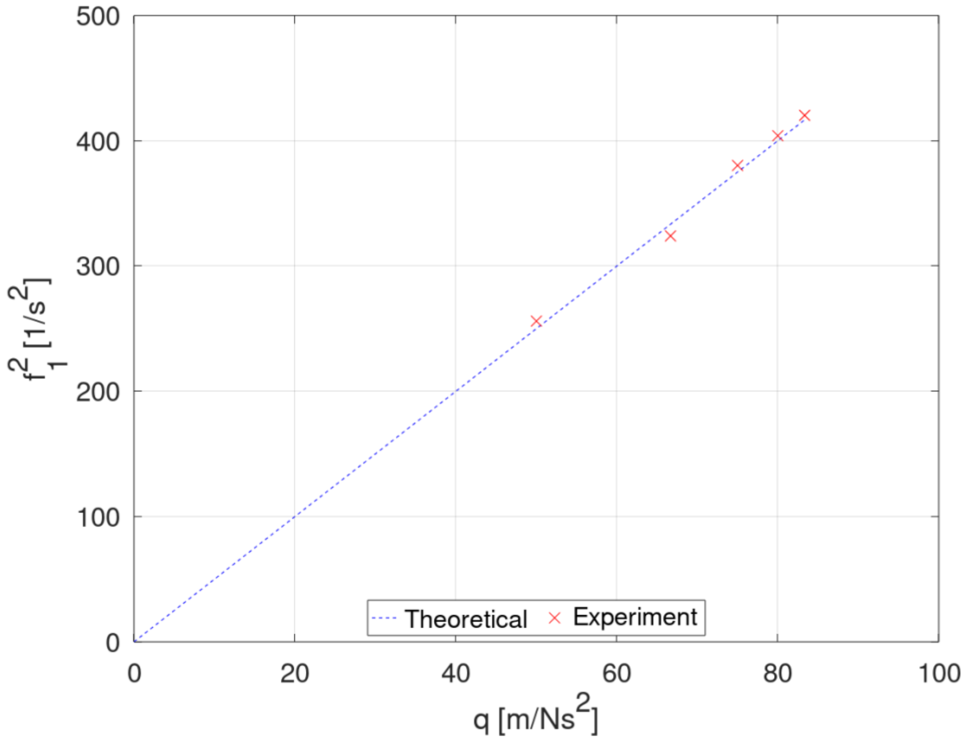

Figure 5: *Plot of* $(f_1)^2$ $vs. q$, *were* $q = \frac{1}{4m}\left(1 - \frac{\hat{L}}{L_0}\right)$. *The experimental data (red x) is from Table 1. The slope of the theoretical fit (blue line) is k = 5 [N/m].*

Once the value of k is found, it is instructive to plot f_1 as a function of L_0 (see Fig 6):

$$f_1 = \sqrt{\frac{k}{4m}\left(1 - \frac{\hat{L}}{L_0}\right)}. \qquad \text{Eq [37]}$$

As shown in Fig 6, students will notice that as L_0 increases the fundamental frequency increases but eventually it flattens out and approaches the constant value:

$$f_1{}^* = \sqrt{\frac{k}{4m}}. \qquad \text{Eq [38]}$$

This means that for strings that behave like *perfectly elastic materials* ($L_0 \gg \hat{L}$), the fundamental frequency is a constant that depends on the material properties of the string k and m, and therefore independent of the initial stretch L_0 and/or tension T_0.

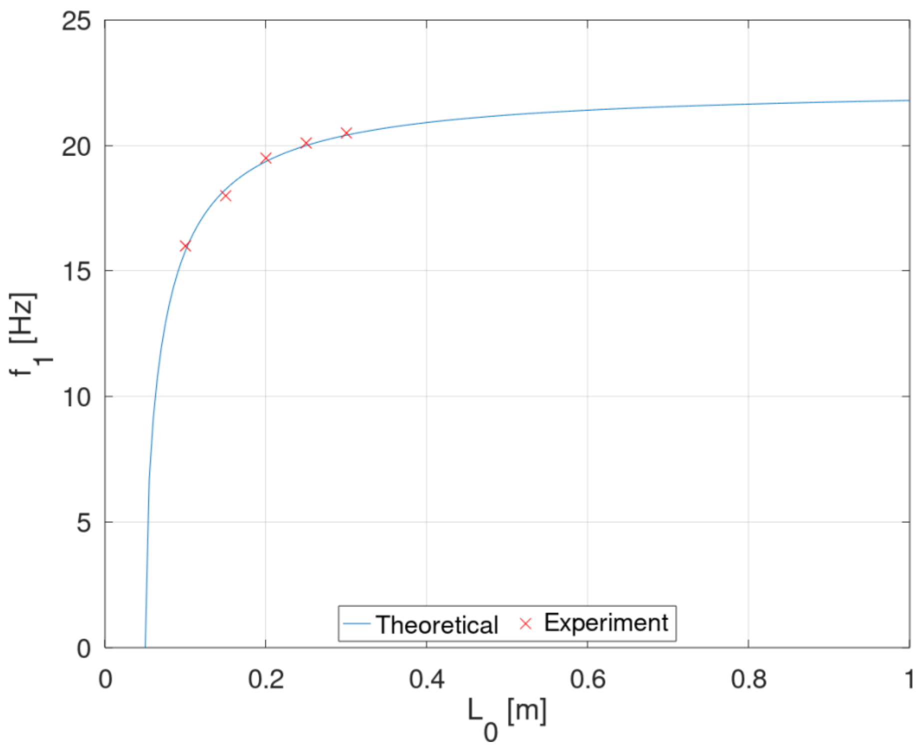

Figure 6: *Plot of f_1 as a function of L_0. The experimental data (red x) is from Table 1. The theoretical fit (blue line) approaches ${f_1}^*$given by Eq [38]*

REFERENCES

Alonso, M., & Finn, E. (1983). *Fundamental university physics vol. 2 : fields and waves.* Addison-Wesley.

Antman, S. S. (1980). The Equations for Large Vibrations of Strings. *American Mathematical Monthly, 87*(5), 359-370.

Caamano Withall, Z., & Krysl, P. (2016). Taut String Model: Getting the Right Energy versus Getting the Energy the right Way. *World Journal of Mechanics, 6*, 24-33. doi:10.4236/wjm.2016.62004

Clelland, J. N., & Vassiliou, P. J. (2013). Strings Attached: New Light on an old problem. *arXiv e-prints, math.AP*(arXiv:1302.6672).

Fowler, M. (n.d.). *Analyzing Waves on a String.* Retrieved 8 24, 2021, from http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/AnalyzingWaves.htm

French, A. (2003). *Vibrations and Waves.* Norton.

Garret, S. L. (2020). String Theory. In *Understanding Acoustics. Graduate Text In Physics.* (pp. 133-178). Springer, Cham. doi:https://doi.org/10.1007/978-3-030-44787-8_3

Giancoli, D. C. (2020). *Physics for Scientists and Engineers with Modern Physics (5th Edition).* Pearson Education (US).

Hyperphysics. (2022). *Energy in a String Wave*. Retrieved 24 8, 2021, from http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/powstr.html

Katz, D. M. (2014). *Physics for Scientists and Engineers: Foundations and Connections.* Cengage Learning.

Keller, J. B. (1959). Large Amplitude Motion of a String. *American Journal of Physics, 27*(8), 584-586.

Knight, R. (2016). *Physics for Scientists and Engineers (4th Edition).* Pearson Education (US).

Lee, R. B., & W., G. a. (1988). *Partial Differential Equations of Mathematical Physics and Integral Equations.* Prentice Hall.

Ling, S. J., Sanny, J., & Moebs, W. (2016). *University Physics Volume 1: OpenStax.*

Luke, J. C. (1992). The motion of a stretched string with zero relaxed length in a gravitational field. *American Journal of Physics, 60*(6), 529-532

Myint U., T., & Debnath, L. (2007). *Linear Partial Differential Equations for Scientists And Engineers* (4th ed.). Birkhauser Boston: Springer.

Ng, C. (2010). Energy in a String Wave. *The Physics Teacher, 48*(1), 46-47.

O'Reilly, O. M. (2017). *Modelling Nonlinear Problems in the Mechanics of Strings and Rods. The role of the balance laws.* Berlekey: Springer.

Ryan, M., Neary, E., RInaldo, J., & Olivia, W. (2019). *Introductory Physics: Building Models to Describe Our World.* doi:10.24908/BPWM9859

Salsa, S. (2008). *Partial Differential Equations in Action (From Modelling to Theory).* Milano: Springer-Verlag Italia.

Serway, R. A., & Jewett, J. W. (2019). *Physics for scientists and engineers with Modern Physics.* Cengage.

Shankar, R. (2014). *Fundamentals of Physics: Mechanics, Relativity, and Thermodynamics.* New Haven and London.

Tipler, P., & Mosca, G. (2007). *Physics for Scientists and Engineers (6th Edition).* Macmillan Higher Education.

Walker, J., Halliday, D., & Resnick, R. (2018). *Fundamentals of Physics, 11th edition.* Wiley.

Wolfson, & Richard. (2011). *Essential University Physics, 2nd Edition.* Addison-Wesley Pearson.

Yong, D. H. (2006). Strings, Chains, and Ropes. *Siam Review, 48*(4), 771-781.

Young, H. D., & Freedman, R. A. (2016). *University Physics: with Modern Physics, 14th Edition.* Pearson.

APPENDIX - Some points where a student might need some clarification

A1: Mass per unit length in the equilibrium vs. deformed configurations.

Students might wonder why we use $\delta m = \mu_o \delta x$, shouldn't we use the length δs of the deformed section and therefore shouldn't we have a mass per unit length $\mu(x,t)$ that depends on how much we stretch the string which depends on space and time?

First, we recognize that the total mass δm will be given by:

$$\delta m = \int_x^{x+\delta x} \mu(x,t)dx. \qquad \text{Eq [A1]}$$

In the equilibrium configuration (Figure 1a), the mass per unit length is uniform $\mu(x,t) = \mu_o$ and hence the integral reduces to the familiar:

$$\delta m = \int_x^{x+\delta x} \mu_o dx = \mu_o \delta x. \qquad \text{Eq [A2]}$$

Recall that the mass δm is a conserved quantity, it does not change even when the string stretches. Therefore $\mu_o \delta x$ is a constant that we can use to represent the mass of an element at any point in time. It might be worth to point out that $\mu(x,t)$ is not conserved and changes as the waves travel through the string. One can find the relationship between $\mu(x,t)$ and μ_o by noticing that in the deformed configuration we will have:

$$\delta m = \int_s^{s+\delta s} \mu(x,t)ds, \qquad \text{Eq [A3]}$$

where $ds = dx\sqrt{1 + y'(x)^2}$ is the infinitesimal stretch. Using a change of variable for the integration the mass in the deformed state is:

$$\delta m = \int_x^{x+\delta x} \mu(x,t)\sqrt{1 + y'(x)^2}\, dx\,. \qquad \text{Eq [A4]}$$

Since δm is conserved quantity, then:

$$\mu(x,t) = \frac{\mu_o}{\sqrt{1 + y'(x)^2}}. \qquad \text{Eq [A5]}$$

A2: Expression for the tension T(x) of the string in the deformed configuration (waves present).

We define the stretch variable λ_o which measures the change in length of the string going from the natural unstretched string state ($\hat{L}$) to the equilibrium stretched state (L_o):

$$\lambda_o = \frac{\delta x}{\delta \hat{x}} = \frac{L_o}{\hat{L}}. \qquad \text{Eq [B1]}$$

Similarly, the stretch variable λ_s measures the extension going from the natural unstretched string state ($\hat{L}$) to the deformed stretched state (L) where a wave is present:

$$\lambda_s = \frac{\delta s}{\delta \hat{x}} = \frac{\delta s}{\delta x}\lambda_o \qquad \text{Eq [B2]}$$

Then we use the definition of the tension $T(x)$ from the main text as a function of the equilibrium tension T_o:

$$\frac{T(x)}{T_o} = \frac{\delta s - \delta \hat{x}}{\delta x - \delta \hat{x}} = \frac{\lambda_s - 1}{\lambda_o - 1} = \frac{\frac{\delta s}{\delta x}\lambda_o - 1}{\lambda_o - 1} = \frac{\delta s}{\delta x}\left(\frac{\lambda_o}{\lambda_o - 1}\right) - \frac{1}{\lambda_o - 1}. \qquad \text{Eq [B3]}$$

Where we have used the definitions of the stretch in the equilibrium and deformed configuration. For the sake of completion, we show below the algebraic steps that lead to the final expression:

$$\frac{T(x)}{T_o} = \frac{\delta s}{\delta x}\left(1 + \frac{1}{\lambda_o - 1}\right) - \frac{1}{\lambda_o - 1} = \frac{\delta s}{\delta x} + \left(\frac{1}{\lambda_o - 1}\right)\left(\frac{\delta s}{\delta x} - 1\right), \qquad \text{Eq [B4]}$$

$$\frac{T(x)}{T_o} = \frac{\delta s}{\delta x} + \left(\frac{\hat{L}}{L_o - \hat{L}}\right)\left(\frac{\delta s}{\delta x} - 1\right). \qquad \text{Eq [B5]}$$

Finally, we can arrive to a more compact expression by using the definition of the engineering stress in the equilibrium configuration $\varepsilon_o = \left(L_o/\hat{L} - 1\right)$ such that the tension is:

$$T(x) = T_o\left[\frac{\delta s}{\delta x}\left(1 + \frac{1}{\varepsilon_o}\right) - \frac{1}{\varepsilon_o}\right]. \qquad \text{Eq [B6]}$$