Degeneracy of 1D-Harmonic Oscillator

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Abstract

Eigenstates of a quantum mechanical system reflect one to one correspondence with energy level. However, in some systems more than one or a group of eigenstates correspond to a single eigenvalue. To introduce more than one eigenstate corresponding to single energy eigenvalue in 1D-harmonic oscillator, we introduce a new perturbation term and find entire eigenspectrum become degenerate in nature without changing the eigenfunctions of the system.

Key words: Degeneracy, one dimensional, harmonic oscillator, differentiation.

INTRODUCTION

In classical physics, a system corresponds to a single energy. However, in quantum mechanics single energy is replaced by energy level corresponding to single eigenstate. Mathematically, there is one to one correspondence between energy level and eigenstate (Bender and Boettcher, 1998; Biswas, 2013; Jorda, 2018; Griffiths, 2005; Gupta et al., 1974; Rath, Mallick, and Mohapatra, 2020) i.e.

$$|\Psi_i\ > \longrightarrow E_i \tag{1}$$

In fact, the no of eigenstates of quantum mechanical systems are very large. A simple example of this is one-dimensional harmonic oscillator having Hamiltonian of the form,

$$H = P^2 + x^2 \tag{2}$$

having eigenfunction

$$\Psi_n = \sqrt{\frac{\sqrt{\pi}}{n!2^n}} H_n(x) e^{-\frac{x^2}{2}} \quad (3)$$

In the above, $H_n(x)$ is the Hermite polynomial , which can be generated as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad (4)$$

Here, the eigenvalues of the system are,

$$E_n = (2n + 1) \quad (5)$$

In above, we see there is a one-to-one correspondence between eigenfunction and energy eigenvalue. However, all the quantum systems do not respond to this one-to-one correspondence. In fact, the eigenstates correspond to a single energy eigenvalue. Mathematically,

$$\overbrace{\Psi_i;\ \Psi_j;\ \Psi_{k\ldots.}} \rightarrow E_n \quad (6)$$

This type of system is called a "degenerate system". Let us consider few simple examples of these type systems as follows.

2D-Harmonic Oscillator

The Hamiltonian of this model is,

$$H = \frac{p_x^2}{2} + \frac{x^2}{2} + \frac{p_y^2}{2} + \frac{y^2}{2} \quad (7)$$

Here, the wave function is a simple product of wave function in x direction and wave function in y-direction i.e.

$$\Psi_{\mathrm{n}}(x, \mathcal{Y}) = \psi_{n_x}(x)\, \psi_{n_y}(\mathcal{Y}) \quad (8)$$

One can see that two different symbols n_x; n_y are related with each other by the linear addition rule i.e.

$$n = n_x + n_y \quad (9)$$

For example,

(i)$n = 0 \longrightarrow n_x = 0; n_y = 0$

(ii)$n = 1 \longrightarrow (n_x = 1; n_y = 0)(n_x = 0; n_y = 1)$

For n=1, we have two different wave functions. Mathematically n=1 is a degenerate state, where as n=0 is non-degenerate in nature. Similarly, all higher states are degenerate in nature. Now, the energy level of this 2D-oscillator is,

$$E_n = (n + 1) \qquad (10)$$

For n=1, $E_2 = 2$ and we have to eigenstates.

3D-Harmonic Oscillator

Consider a three-dimensional Harmonic oscillator Hamiltonian as,

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{p_z^2}{2} + \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \qquad (11)$$

having energy eigenvalue

$$E_n = n + \frac{3}{2} \qquad (12)$$

where $n = n_x + n_y + n_z$. The corresponding wave function is,

$$\Psi_n(x, \mathcal{Y}, z) = \psi_{n_x}(x)\, \psi_{n_y}(\mathcal{Y})\ \psi_{n_z}(z) \qquad (13)$$

For n=2, we have 6 no of wave functions as $\psi_{2,0,0}$; $\psi_{0,2,0}$; $\psi_{0,0,2}$; $\psi_{1,1,0}$; $\psi_{0,1,1}$; $\psi_{1,0,1}$. In fact, it a 6-fold degenerate system. In general, interested reader will notice that total no of wave functions for a given energy is $\frac{(n+1)(n+2)}{2}$.

Two – Dimensional Box

Here, we consider a two-dimensional box having side "a". Potential in this system can be expressed as

$$V\,(x, \mathcal{Y}) = 0{,}0 \ll x \ll a; 0 \ll \mathcal{Y} \ll a \qquad (14)$$

The wave function of the system is,

$$\Psi_{nx,ny}(x, \mathcal{Y}) = \psi_{n_x}(x)\, \psi_{n_y}(\mathcal{Y}) \tag{15}$$

Where,

$$\psi_{n_x}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \tag{16}$$

$$\psi_{n_y}(\mathcal{Y}) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi \mathcal{Y}}{a}\right) \tag{17}$$

Energy level of the system is,

$$E_n = \frac{h^2}{8ma^2} n^2 \tag{18}$$

where $n^2 = n_x^2 + n_{\mathcal{Y}}^2$. For $n^2 = 5$, we have (1,2) ; (2,1). This means, there is a twofold degeneracy in the system. Here, the ground state is non-degenerate having energy,

$$E_2 = \frac{h^2}{4ma^2} \to \Psi_{1,1}(x, \mathcal{Y}) \tag{19}$$

Three-Dimensional Box

As above in a three-dimensional box, we have the potential of the form,

$$V\,(x, \mathcal{Y}, z) = 0{,}0 \ll x \ll a; 0 \ll \mathcal{Y} \ll a; 0 \ll z \ll a \tag{20}$$

The wave function of the system is,

$$\Psi_{n_x, n_y, n_z}(x, \mathcal{Y}, z) = \psi_{n_x}(x)\, \psi_{n_y}(\mathcal{Y})\, \psi_{n_z}(z) \tag{21}$$

Where,

$$\psi_{n_x}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \tag{22}$$

$$\psi_{n_y}(\mathcal{Y}) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi \mathcal{Y}}{a}\right) \tag{23}$$

$$\psi_{n_z}(z) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_z \pi z}{a}\right) \tag{24}$$

Energy level of the system is,

$$E_n = \frac{h^2}{8ma^2} n^2 \tag{25}$$

Where $n^2 = n_x^2 + n_y^2 + n_z^2$. For $n^2 = 27$, we have (5,1,1); (1,5,1); (1,1,5); (3,3,3). This means, there is a fourfold degeneracy in the system. Here, the ground state is no-degenerate having energy,

$$E_3 = \frac{3h^2}{8ma^2} \rightarrow \Psi_{1,1,1}(x,y,z) \quad (26)$$

Hydrogen Atom

$$H = \frac{p^2}{2} - \frac{1}{r} \quad (27)$$

The energy level of the system is,

$$E_n = -\frac{1}{2n^2} \quad (28)$$

Further, wave function of the system is Ψ_{nlm} . For a given n, the total no of degeneracy of the system is n^2. For example, n=2, there are 4 no wave functions i.e (2,0,0); (2,1,0); (2,1,1) ;(2,1,-1). In above we notice that degeneracy is a common feature in higher dimension. However, here we discuss degeneracy in 1D-harmonic oscillator as follows under the influence of a perturbation.

Harmonic Oscillator Under Complex Perturbation

Interesting feature of the harmonic oscillator is that its wave function is simple to handle in practice. For example,

$$|\,\Phi_0 >= \frac{1}{\pi^{1/4}} e^{-x^2/2} \quad (29)$$

Let us calculate $\lambda = \pm 1$, using HO low level wave functions as,

$$< \Phi_0 | -\frac{i}{x} p = -\frac{1}{x}\frac{d}{dx} | \Phi_0 > = 1 \quad (30)$$

Similarly, using suitable recurrence relation [2], one will find,

$$< \Phi_1 | -\frac{i}{x} p | \Phi_1 > = -1 \quad (31)$$

If this term can be added to $E_{n,}$ we find that,

$$H_D = H - \frac{i}{x} p \quad (32)$$

$$< \Phi_n | H_D | \Phi_n > 0,1 = 2 \quad (33)$$

In fact, we have made eigenvalues of ground state and first excited state are the same, even though they possess different eigenfunctions. Now, extending this to higher excited states we find.

$$< \Phi_n| -\frac{i}{x}p\,| \Phi_n >= -< \Phi_n| \frac{1}{x}\,| \Phi'_n >= \pm 1 \qquad (34)$$

Here the +1 sign for even states including 0 i.e., n=0,2,4,6,8…and -1 for n=1,3,5,7…… Now, combining this we get

$$< \Phi_n| - H_D\,| \Phi_n >= \epsilon_n = 2,6,10,14\ldots \qquad (35)$$

More explicitly above relation is to be interpreted as

$$< \Phi_i| H_D| \Phi_i >=< \Phi_{i+1}| H_D| \Phi_{i+1} >= \epsilon_i \ (i = 0,2,4) \qquad (36)$$

DISCUSSION

In nature, we come across twins, which are indistinguishable in their physical feature. In Mathematics, we also come across twin number such as (2,2). Using simple algebra, one can generate set of twin numbers as follows. Suppose we have all positive odd integers such as 1,3,5,7,9………. In this numbers we find that even numbers are missing. Now we want that entire odd numbers to be converted to even number such that adjacent numbers will have the same value. Then simple way is to add +1 to lower number and subtract the same from the next higher number. In this procedure we will get (2,2) ; (6,6) ; (10,10); (14,14) so on so. This is done with elementary algebra of addition and subtraction. The above mathematical analysis is related to a physical problem of simple oscillator using the method of integration and differentiation as discussed above. From physics point of view, we find that a simple perturbation $\left(\frac{-i}{x}p\right)$ in 1D harmonic oscillator can reflect degeneracy. In fact, all even state eigenvalue is increased by +1 and odd state eigenvalue decreased by -1. As a result, two consecutive levels reflect the same eigenvalue. In other words, eigenvalues of H_D are (2,2); (6,6); (10,10), (14,14). It should be borne in mind that the perturbation terms are PT-symmetry in nature [5,6]. In fact, behavior of Parity (P) in a quantum system is as follows:

$PxP^{-1} = -x; P_pP^{-1} = -p; PiP^{-1} = i$. Similarly, T (time-reversal) operator has the following behavior $TiT^{-1}; T_pT^{-1} = -p; T_xT^{-1} = x$. Hence interested reader will notice that the perturbation term $H_1 = \frac{-i}{x}p$ is PT—symmetric. Mathematically, $[H_1, PT] = 0$. Hence, it is not difficult to show that the Hamiltonian is PT-symmetric in nature i.e., $[H, PT] = 0$. Interested reader will notice that simple integration and differentiation can change basis understanding of quantum mechanics i.e., from non-degenerate to degenerate with the aid of suitable simple term H_1.

The method suggested here can be extended to few oscillatory systems having the form of Hamiltonians as

$$H = p^2 + x^2 + x^{2m} \rightarrow m = 2,3, \dots \quad (37)$$

Interested readers will see that under the influence of perturbation H_1, the system will reflect degeneracy which gives the importance of generating degenerate oscillatory system.

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