Generalization of the Cosines Theorem to Polygons and Its Application to the Analytical Calculation of Multi-Beam Interference of Coherent Radiation of Light with an Arbitrary Phase Distribution

Alexandra Glushchenko

Alexander Glushchenko

Eugenia Glushchenko

Volga State University of Telecommunications and Informatics of Samara

Samara, Samara Oblast, Russia

gag646@yandex.ru

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Abstract

The cosine theorem is used in solving triangulation problems and in physics when solving problems of addition of unidirectional oscillations. However, this theorem is used only for the analytical calculation of triangles or when solving problems of adding two oscillations. Here we propose a generalization of the cosine theorem for the case of calculating polygons of arbitrary shape and consider its application for calculating the results of adding an arbitrary number of unidirectional oscillations or coherent waves with an arbitrary phase distribution.

Keywords**:** polygons, generalized cosine theorem, addition of oscillations, addition of coherent waves

INTRODUCTION

Special attention is always paid to the possibility of obtaining accurate analytical solutions of mathematical models of physical processes. Analytical solutions allow an accurate analysis of the properties of the objects under consideration. In mathematics, a large role is given to the cosine theorem (Khyfits,2004,Pickover,2009,Rade,2004), which allows one to obtain an analytical solution to one particular, but practically important problem of triangulation: to calculate one of the sides of a triangle from the known two sides of the triangle and the angle between them. This theorem is also often used in problems of the theory of oscillations and waves to find the result of the addition of two unidirectional oscillations (Hewitt,2015,Randall,2016), or coherent waves (Thorne,2016)

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of the same polarization. In this paper, we show that the cosine theorem can be essentially generalized to convex and non-convex polygons with any number of sides: any side of a polygon can be determined from the known other sides of the polygon and the angles between them. This theorem also allows one to obtain new analytical solutions to an important problem for optics and technology of wave systems: calculating the result of adding an arbitrary number of discrete (Glushchenko,2016) and continuous distributions of sources of coherent waves with an arbitrary distribution of the phase delay function.

Basic Results

Consider a triangle with sides and an angle between these sides of the triangle (Fig. 1). Notation is used here in a form convenient for generalization.

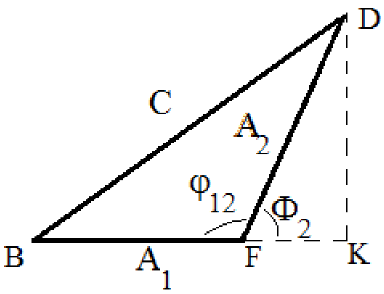

Figure 1. *Triangle with given sides* A_1, A_2 *and the angle between them* φ_{12}

The cosine theorem states that side C of the triangle BDF can be found from the relation:

$$C^2 = A_1^2 + A_2^2 - 2A_1A_2\cos\varphi_{12}\,. \tag{1}$$

We represent this relation in an equivalent, convenient for further generalization, form of writing:

$$C^2 = \left(A_1 + A_2\cos\Phi_2\right)^2 + \left(A_2\sin\Phi_2\right)^2 \tag{2}$$

It follows from Fig. 1 that the desired side C is the hypotenuse of the right triangle BDK, built on the basis of the given triangle BDF by extending the side BF and lowering the perpendicular to the line BK. Equation (2) is thus a record of the Pythagorean Theorem for the triangle BDF. Expressing in (2) Φ_2 through the angle φ_{12} between the known parties A_1, A_2 we have:

$$C^2 = \{A_1 + A_2\cos(\pi - \varphi_{12})\}^2 + \{A_2\sin(\pi - \varphi_{12})\}^2 \tag{3}$$

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A similar construction for a quadrangle (Fig. 2) with given sides A_1, A_2, A_3 and angles $\varphi_{12}, \varphi_{23}$ between them allows for the right triangle BDK to represent the hypotenuse C of this triangle in the form:

$$C^2 = (A_1 + A_2\cos\Phi_1 + A_3\cos\Phi_2)^2 + (A_2\sin\Phi_1 + A_3\sin\Phi_2)^2 =$$
$$= \{A_1 + A_2\cos[\pi - \varphi_{12}] + A_3\cos[2\pi - (\varphi_{12} + \varphi_{23})]\}^2 +$$
$$+ \{A_2\sin[\pi - \varphi_{12}] + A_3\sin[2\pi - (\varphi_{12} + \varphi_{23})]\}^2$$

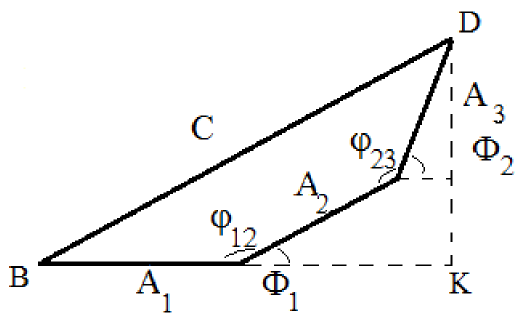

Figure 2. Quadrilateral with given sides A_1, A_2, A_3 and the angles between them $\varphi_{12}, \varphi_{23}$

We represent this relation in a form similar to (3)

$$C^2 = \{A_1 + A_2\cos[\pi - \varphi_{12}] + A_3\cos[2\pi - (\varphi_{12} + \varphi_{23})]\}^2 + \{A_2\sin[\pi - \varphi_{12}] + A_3\sin[2\pi - (\varphi_{12} + \varphi_{23})]\}^2 \quad (4)$$

Similar constructions for a polygon with (4) known sides A_1, A_2, A_3, A_4 and angles $\varphi_{12}, \varphi_{23}, \ldots, \varphi_{34}$ between them (Fig. 3) allow us to generalize the cosine theorem known in triangles (presented in form (3)) to a polygon. The unknown side C is sought through given sides A_1, A_2, A_3, A_4 in the form of a relation:

$$C^2 = (A_1 + A_2\cos\Phi_1 + A_3\cos\Phi_2 + A_4\cos\Phi_3)^2$$
$$+ (A_2\sin\Phi_1 + A_3\sin\Phi_2 + A_4\sin\Phi_3)^2 =$$
$$= \{A_1 + A_2\cos[\pi - \varphi_{12}] + A_3\cos[2\pi - (\varphi_{12} + \varphi_{23})] + A_4\cos[3\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34})]\}^2$$
$$+ \{A_2\sin[\pi - \varphi_{12}] + A_3\sin[2\pi - (\varphi_{12} + \varphi_{23})] + A_4\sin[3\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34})]\}^2$$

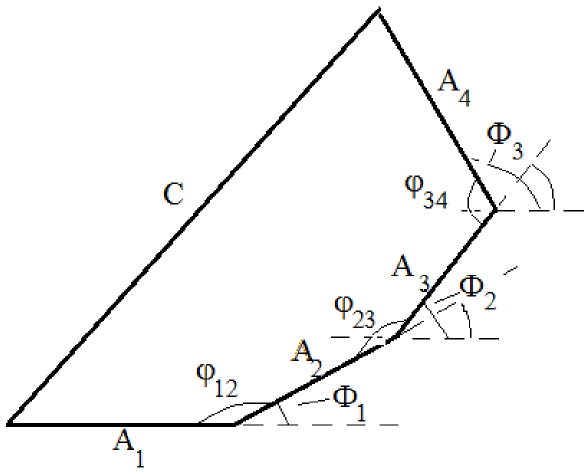

Figure 3. *The polygon*

For a polygon with the number of sides n + 1, the ratio takes the form:

$$C^2 = \{A_1 + A_2 \cos[\pi - \varphi_{12}] + ... + A_n \cos[(n-1)\pi - (\varphi_{12} + \varphi_{23} + ... + \varphi_{n-1,n})]\}^2 + \\ + \{A_2 \sin[\pi - \varphi_{12}] + ... + A_n \sin[(n-1)\pi - (\varphi_{12} + \varphi_{23} + ... + \varphi_{n-1,n})]\}^2 \tag{5}$$

where A_i $(i = 1,...,n)$ are the known sides of the polygon, C is the desired side, and $\varphi_{12}, \varphi_{23},...,\varphi_{n-1,n}$ are the angles between the known adjacent sides.

Calculation Examples

a) Consider a rectangle. On the three sides $A_1 = 2, A_2 = 3, A_3 = 2$ and the angles $\varphi_{12} = \varphi_{23} = \pi/2$ between them, we are looking for the fourth side according to the formula (4).Then

$$C^2 = \{2 + 3\cos[\pi - \pi/2] + 2\cos[2\pi - (\pi)]\}^2 + \{3\sin[\pi - \pi/2] + 2\sin[2\pi - (\pi)]\}^2 = 3^2,$$
$$C = 3$$

b) Consider an isosceles trapezoid: $A_1 = 2, A_2 = 2, A_3 = 2, \varphi_{12} = \varphi_{23} = 2\pi/3$

We are looking for the trapezoid base according to (4)

$$C^2 = \left(2 + 2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3}\right)^2 + \left(2\sin\frac{\pi}{3} + 2\sin\frac{2\pi}{3}\right)^2 = 4^2$$

In this way C=4.

c) for a regular hexagon on five sides and the corners between them the sixth side is equal

$$A_1 = A_2 = A_3 = A_4 = A_5 = 2,\ \varphi_{12} = \varphi_{23} = \varphi_{34} = \varphi_{45} = \frac{2\pi}{3},$$

$$C^2 = \left\{\begin{matrix} A_1 + A_2\cos[\pi - \varphi_{12}] + A_3\cos[2\pi - (\varphi_{12} + \varphi_{23})] + \\ + A_4\cos[3\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34})] + A_5\sin[4\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34} + \varphi_{45})] \end{matrix}\right\}^2$$
$$+ \left\{\begin{matrix} A_2\sin[\pi - \varphi_{12}] + A_3\sin[2\pi - (\varphi_{12} + \varphi_{23})] + \\ + A_4\sin[3\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34})] + A_5\sin[4\pi - (\varphi_{12} + \varphi_{23} + \varphi_{34} + \varphi_{45})] \end{matrix}\right\}^2 =$$
$$= \left\{2 + 2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3} + 2\cos\pi + 2\cos\frac{4\pi}{3}\right\}^2 + \left\{2\sin\frac{\pi}{3} + 2\sin\frac{2\pi}{3} + 2\sin\pi + 2\sin\frac{4\pi}{3}\right\}^2,$$
$$C = 2$$

Thus, formula (5) can be used to calculate the sides of any polygon with any angles between adjacent sides (Fig. 4).

d) Consider the quadrangle shown in Fig. 4*a*.

$$A_1 = 6,\ A_2 = 2\sqrt{3},\ A_3 = 2\sqrt{3},\ \varphi_{12} = \pi/6, \varphi_{23} = 2\pi/3.$$

We are looking for the trapezoid base according to (4)

$$C^2 = \{A_1 + A_2\cos[\pi - \varphi_{12}] + A_3\cos[2\pi - (\varphi_{12} + \varphi_{23})]\}^2 +$$
$$+ \{A_2\sin[\pi - \varphi_{12}] + A_3\sin[2\pi - (\varphi_{12} + \varphi_{23})]\}^2 =$$
$$= \left\{6 + 2\sqrt{3}\cos\frac{5\pi}{6} + 2\sqrt{3}\cos\frac{\pi}{2}\right\}^2 + \left\{2\sqrt{3}\sin\frac{5\pi}{6} + 2\sqrt{3}\sin\frac{\pi}{2}\right\}^2 = 6^2,$$
$$C = 6$$

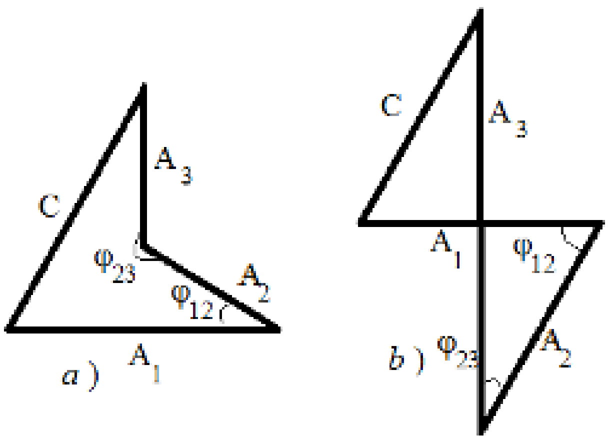

Figure 4. *Non-convexpolygons*

f) Consider the quadrangle shown in Fig. 4b.

$$A_1 = 6,\ A_2 = 6,\ A_3 = 3\sqrt{3},\ \varphi_{12} = \pi/3, \varphi_{23} = \pi/6$$

Looking for side C by (4)

$$C^2 = \left\{6 + 6\cos\frac{2\pi}{3} + 3\sqrt{3}\cos\frac{3\pi}{2}\right\}^2 + \left\{6\sin\frac{2\pi}{3} + 6\sqrt{3}\sin\frac{3\pi}{2}\right\}^2 = 6^2,$$

$C = 6$

Thus, relation (4) can be used not only for convex polygons, but for polygons of any configuration.

Application in the theory of interference

Let us consider the application of the generalized cosine theorem to the calculation of the problem of adding waves of coherent sources that often arises in optics, which are described by the functions:

$$E_1(\boldsymbol{r},t) = A_1 \, cos(\omega_1 t - \boldsymbol{k}_1\boldsymbol{r}_1 + \varphi_1) = A_1 \, cos\, \Phi_1,$$

$$\ldots$$

$$E_i(\boldsymbol{r},t) = A_i \, cos(\omega_i t - \boldsymbol{k}_i\boldsymbol{r}_i + \varphi_i) = A_i \, cos\, \Phi_i,$$

where A_i are the amplitudes, $\Phi_i = \omega_i t - \boldsymbol{k}_i\boldsymbol{r}_i + \varphi_i$ are the phases, ω_i are the cyclic frequencies, $\boldsymbol{k}_i$ are the wave vectors, $\boldsymbol{r}_i$ are the radius vectors connecting the wave sources and the observation point, φ_i are the initial phases, $i = 1,2,\ldots,n$ of the summed waves at the observation point. The addition of a large number of oscillations at the observation point is carried out using the vector diagram shown in Fig. 5.

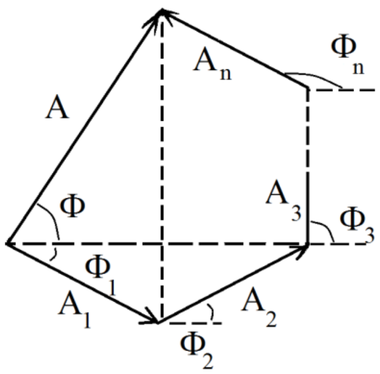

Figure 5. *Multipath interference*

The amplitude of the resulting oscillation depends on the amplitudes and phases of the added waves

$$\begin{aligned}A^2(\boldsymbol{r},t)&=(A_1\cos\Phi_1+\ldots+A_n\cos\Phi_n)^2+(A_1\sin\Phi_1+\ldots+A_n\sin\Phi_n)^2=\\&=A_1^2+A_2^2+\ldots+A_n^2+2A_1A_2\cos(\Phi_1-\Phi_2)+\ldots+2A_iA_j\cos(\Phi_i-\Phi_j)+\\&+\ldots+2A_{n-1}A_n\cos(\Phi_{n-1}-\Phi_n)=\sum_i^n\sum_j^n A_{ij}\cos(\Phi_i-\Phi_j)\end{aligned}\qquad(6)$$

phase is determined by the ratio:

$$\Phi=arctg\,\frac{A_1\sin\Phi_1+A_2\sin\Phi_2+\ldots+A_n\sin\Phi_n}{A_1\cos\Phi_1+A_2\cos\Phi_2+\ldots+A_n\cos\Phi_n}=\frac{\sum_i^n A_i\sin\Phi_i}{\sum_i^n A_i\cos\Phi_i}$$

The obtained relationships can be used to "tune" the required radiation level at the observation point using phase shifters. For example, for two emitters from relation (6), which we represent in the form:

$$A^2(\boldsymbol{r},t)=A_1^2+A_2^2+2A_1A_2\cos[k(r_1-r_2)+\varphi_1-\varphi_2]$$

the necessary difference of the initial phases can be found for the realization of any radiation level for any point in space. The required phase difference is determined by the ratio:

$$\varphi_1-\varphi_2=k(r_2-r_1)+arccos\,\frac{A^2-A_1^2-A_2^2}{2A_1A_2}+2\pi m,\quad m=0,1,2,\ldots$$

In practice, this condition can be realized using a single phase shifter. With a larger number of emitters, the total radiation level can also be controlled by phase shifters through which individual emitters are excited.

The structure of the radiation intensity for three coherent emitters in two configurations is shown in Fig. 6(a,b)

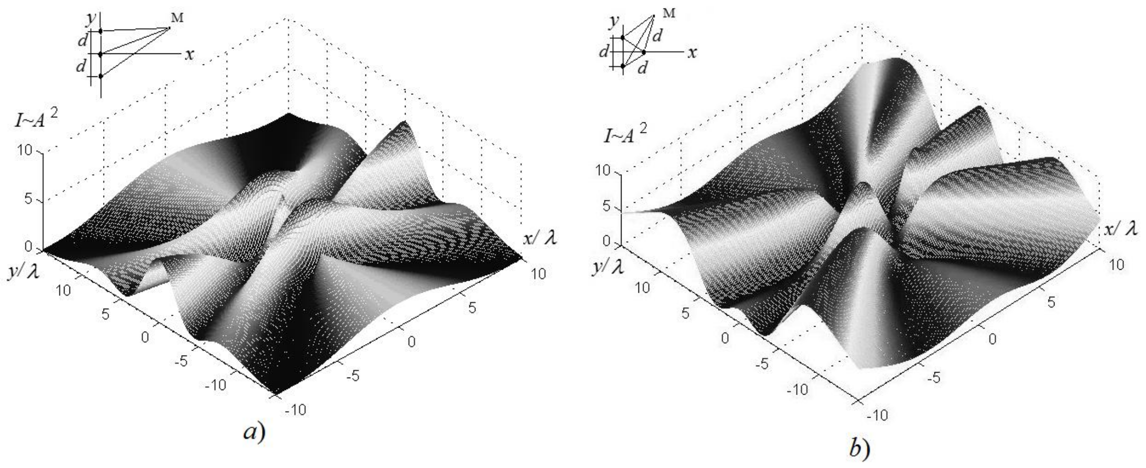

Figure 6. *Spatial distribution of radiation intensity for three emitters of two configurations*

The radiation intensity distributions shown in the figures show that changing the configuration of the arrangement of point emitters, their number, and controlling the difference in their initial phases allows changing the radiation pattern within wide limits. The relations obtained make it possible to calculate the radiation patterns of any number of arbitrarily located emitters with any initial phases in isotropic and anisotropic media.

CONCLUSION

A generalization of the cosine theorem for triangles to the case of polygons is carried out, which allows one to analytically calculate the sides of convex and non-convex n gons for given n-1 sides of polygons and the angles between them. The obtained relations can be used in optics and physics of wave processes for practical problems of analytical calculation of amplitudes and phases of signals as a result of multibeam interference of discrete light sources or emitters of waves of a different physical nature. It should be noted that the results obtained can be generalized to the case of polygons with non-rectilinear sides, which corresponds to the problem of continuous source distribution, which is typical for problems of light diffraction.

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