CALCULATION OF MINIMUM SPEED OF PROJECTILES UNDER LINEAR RESISTANCE USING THE GEOMETRY OF THE VELOCITY SPACE

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Abstract

We look at the problem of the minimum speed of projectiles in a constant gravitational field. In the absence of resistance, the problem may be studied in the frame of a high school curriculum. One needs only Newton's laws and a minimum amount of analytic geometry to compute the orbit, which turns out to be parabolic. Furthermore, in case the projectile is launched upwards, employing the theorem of conservation of mechanical energy we conclude that the minimum speed occurs at the highest point. In the presence of resistance, the system is dissipative and hence, the previous tools are not available. We focus at the case where the resisting force is linear in speed and opposing the velocity vector. It has been detected in numerical experiments that the minimum speed, if any, occurs when the projectile is on the way down, after having achieved the maximum height. We propose a presentation of the solution to this problem using the geometry of the velocity space. As a tool, the old idea of the hodograph of motion introduced by Hamilton and Maxwell is applied. It turns out that the hodograph of motion in this case is a straight line. This fact allows us to describe the values of initial speed and launching angle that will result in an orbit with or without minimum speed. In the former case, the calculation of the value of minimum speed represents the distance of the origin to the hodograph line and is calculated by elementary manipulations. This approach in the study of physical problems, besides being elegant on its own right, helps college students feel the deep relation between physics and geometry.

Keywords: Projectile motion, linear resistance, hodograph of motion.

INTRODUCTION

Classical mechanics is a continuous source of inspiration for mathematics. An all-time favorite example is projectile motion. In the absence of resistance, the orbit is a parabolic and there are two cases of interest. When the projectile is launched upwards, we can employ the theorem of conservation of mechanical energy to conclude that the minimum speed will occur at the maximum point of the orbit. In the case that the projectile is launched horizontally or downwards, there is no minimum since the speed is increasing for all time.

The next step is to introduce tangential resistance opposing the motion. It has been observed, and verified numerically in (Miranda, Nikolskaya & Riba, 2004), that when the magnitude of the resistance is proportional to some positive integer power of the speed, the minimum speed (if any) occurs after the maximum height position, when the projectile is on the way down. In this work we set the task to investigate and prove this curious fact in the case of linear air resistance. The method of study is geometric, employing the hodograph of motion. We calculate the minimum speed and characterize the type of initial data (initial speed and angle of inclination) which allow the minimal speed to occur.

The organization of the paper is as follows. In the second paragraph we set the notation and establish the closed form solution. In paragraph three we study the geometry of the velocity space. We discover that the hodograph of motion is a straight-line segment. Furthermore, the critical points for the speed occur on a certain semicircle, whose diameter equals the terminal speed. In paragraph four we establish, the physically and geometrically evident fact that critical points for the speed correspond to minima. Having established the geometric picture, it is easy to understand in paragraph five in which cases speed achieves minimum and why it occurs after the maximum height. Furthermore, we can give a simple calculation for its value. Conclusions follow in paragraph six.

The Linear Resistance Model

We choose Cartesian coordinates (x, z), where x is horizontal, and z is vertical. The corresponding basic unit vectors are $\hat{i}$ and $\hat{k}$. The position vector of projectile P in time t is described by

$$\vec{r} = \big(x(t), y(t)\big) \tag{1}$$

Consequently, its velocity, acceleration and speed are given, respectively, by

$$\vec{v} = \big(\dot{x}(t), \dot{y}(t)\big), \quad \frac{d\vec{v}}{dt} = \big(\ddot{x}(t), \ddot{y}(t)\big), \quad v = \sqrt{\dot{x}^2 + \dot{y}^2} \tag{2}$$

The projectile is launched at time $t = 0$ from the origin $(0, 0)$, forming with the horizontal angle $-\frac{\pi}{2} < \theta_0 < \frac{\pi}{2}$ and with initial velocity

$$\vec{v}_0 = (v_0\, cos\theta_0, v_0\, sin\theta_0\ ) \tag{3}$$

The set of forces acting on P consist of the vertical weight and the opposing air resistance given by the laws

$$\vec{W} = -mg\vec{k}\,, \quad \vec{R} = -mb\,\vec{v} \tag{4}$$

Here m is the mass, g the acceleration of gravity presumed constant, and b is a proportionality factor. Vertical orbits occurring at $\theta_0 = \pm\frac{\pi}{2}$ are elementary special cases. Newton's second law of motion leads to the vector differential equation

$$\frac{d\vec{v}}{dt} = -g\vec{k} - b\vec{v} \tag{5}$$

In coordinate form, equation (5) is written

$$\begin{array}{cc} \ddot{x} = -b\dot{x} & \ddot{z} = -b\dot{z} - g \\ x(0) = 0 & z(0) = 0 \\ \dot{x}(0) = v_0 cos\theta_0 & \dot{z}(0) = v_0 sin\theta_0 \end{array} \tag{6}$$

It is evident that the motion takes place in the vertical *x*-*z* plane. One usually sets,

$$\ddot{z} = 0, \qquad \dot{z} = -v_T$$

to get immediately the downward terminal speed,

$$v_T = \frac{g}{b} \tag{7}$$

The derivation of an explicit solution of (6) is a straightforward exercise (Rees, 1920).

$$\dot{x}(t) = v_0\, cos\theta_0\, e^{-bt} \qquad \dot{z}(t) = -v_T + (v_0\, sin\theta_0 + v_T)\, e^{-bt}$$

$$x(t) = \frac{v_0\, cos\theta_0}{b}\left(1 - e^{-bt}\right) \tag{8}$$

$$z(t) = \frac{v_0\, sin\theta_0 + v_T}{b}\left(1 - e^{-bt}\right) - v_T\, t$$

We observe,

$$\lim_{t\to\infty} x(t) = \frac{v_0 cos\theta_0}{b} \qquad \lim_{t\to\infty} z(t) = -\infty \qquad \lim_{t\to\infty} \vec{v}(t) = \left(0, -\frac{g}{b}\right) \tag{9}$$

Therefore, $x = \dfrac{v_0 cos\theta_0}{b}$ is a vertical asymptote and the vertical downward limiting speed (or terminal speed) equals $v_T = \dfrac{g}{b}$, in accordance with (7).

We remark, that even though we have achieved explicit solutions, we cannot obtain direct answers to basic questions. For example, the calculation of the horizontal range and time of flight requires Lambert's W function (Packel & Yuen, 2004), (Stewart, 2011). For the calculation of the minimum speed one has to invest a substantial amount of effort. The next paragraph offers a geometric method to tackle this last problem.

Hodograph of Motion

Let us be reminded that the hodograph of motion is the locus on the velocity diagram of the head of the velocity vector when its tail is kept at the origin. This idea was introduced by (Hamilton, W. R. 1847), studied by (Maxwell, 1952) and exploited recently, among others, by (Apostolatos, 2003).

In our case, we proceed as follows. By elimination of the factor e^{-bt} in (8) we find,

$$\dot{z} = \frac{v_0\, sin\theta_0 + v_T}{v_0\, cos\theta_0}\dot{x} - v_T \qquad (10)$$

This is an equation of a straight-line segment in the velocity space equipped with Cartesian coordinates $(\dot{x}, \dot{z})$. The endpoints, for $0 \leq t < \infty$, correspond to the initial data $A(v_0\, cos\theta_0, v_0\, sin\theta_0)$ and the terminal speed $T(0, -v_T)$. In the language of dynamical systems, the point $T(0, -v_T)$ is an asymptotically stable equilibrium point for (6), as shown in the following figure.

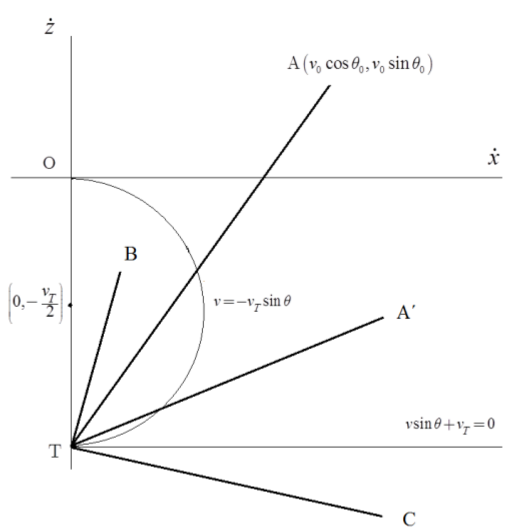

Figure 1. *The line segments AT, A′T, BT, CT represent the hodograph of motion for various initial values of* v_0, θ_0

Next, we characterize the critical points for the speed function. Differentiating $v^2 = \dot{x}^2 + \dot{z}^2$ and employing (6), we obtain after some manipulation

$$-\frac{v}{b}\frac{dv}{dt} = \dot{x}^2 + \left(\dot{z} + \frac{v_T}{2}\right)^2 - \frac{v_T^2}{4} \qquad (11)$$

Therefore, since $v, \dot{x} > 0$, the critical points of the speed where $\frac{dv}{dt} = 0$ occur exactly when $\dot{x}^2 + \left(\dot{z} + \frac{v_T}{2}\right)^2 = \frac{v_T^2}{4}$, that is on the semicircle to the right of the axis, centered at $\left(0, -\frac{v_T}{2}\right)$ and radius $\frac{v_T}{2}$. The corresponding polar equation is of (11) is,

$$v = -v_T \sin\theta\,, \qquad -\frac{\pi}{2} \le \theta < 0 \qquad (12)$$

We finally remark that the horizontal line through $(0, -v_T)$ is given by the equation

$$v_T + v\sin\theta = 0, \qquad -\pi < \theta < 0 \qquad (13)$$

$\dot{z}$

CRITICAL POINTS FOR THE SPEED FUNCTION

Now we study critical points for the speed function. A combination of calculus and physics produces theorem 1 below. We begin with a lemma.

Lemma 1: *Critical points for the speed function v occur exactly when the velocity vector* $\vec{v}$ *is perpendicular to the acceleration* $\frac{d\vec{v}}{dt}$

Proof: Taking into account $v > 0$, the result follows from the well-known calculation

$$\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{d(\vec{v} \cdot \vec{v})}{dt} = \frac{dv^2}{dt} = 2v\,\frac{dv}{dt}$$

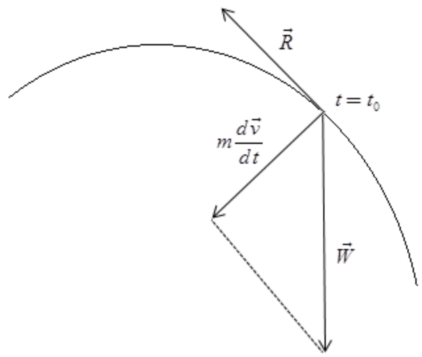

Figure 2. *The geometry and physics of the orbit at a critical point for the speed*

Theorem 1: *Critical points for the speed function can occur only when the projectile is on the way down and correspond to minima.*

Proof: Differentiate $v^2 = \dot{x}^2 + \dot{z}^2$ and use (6), to conclude

$$v\,\frac{dv}{dt} + g\,\dot{z} = -b\,v^2 \tag{14}$$

At a critical point $\frac{dv}{dt}(t_0) = 0$. Hence, $\dot{z} = -\frac{b\,v^2}{g} = -\frac{v^2}{v_T} < 0$ and the projectile is on the way down.

To continue, differentiate (14), employing $\frac{dv}{dt}(t_0) = 0$ and (6) and obtain

$$v\,\frac{d^2v}{dt^2} = g\,(b\,\dot{z} + g) \tag{15}$$

Since $\dot{z} = v\,\sin\theta$, the last relation may be written as

$$v\,\frac{d^2v}{dt^2} = g\,(b\,v\,\sin\theta + g) \tag{16}$$

By lemma 1, at a critical point the $\vec{W}, \vec{R}$ forces acting on the projectile and the

$\frac{d\vec{v}}{dt}$ $\qquad |\vec{W}| > |\vec{R}| > -|\vec{R}|\,sin\theta$

vector form a right triangle. Therefore,

By the first part of the proof, $sin\theta < 0$. This means $mg > -mbv\, sin\theta$ and $g + bv\, sin\theta > 0$. From (16) we have

$$\frac{d^2v}{dt^2} = \frac{g}{v}\,(b\, v\, \sin\theta + g) > 0 \tag{17}$$

By the second derivative test, we conclude $v(t_0)$ is minimum.

Corollary 1. *The speed function cannot have more than one critical points.*

Proof: Assume $\frac{dv}{dt}(t_1) = \frac{dv}{dt}(t_2) = 0$, $t_1 < t_2$. Then for some, $t_1 < t_0 < t_2$, $v(t_0)$ will be maximum, which implies $\frac{dv}{dt}(t_0) = 0$. The last equality contradicts the theorem above.

THE PROBLEM OF MINIMUM SPEED

We now employ geometry. Examining the velocity diagram in Fig. 1, we find that a critical point will occur exactly when the hodograph of motion intersects the semicircle $v = -v_T\, sin\theta$

At any moment, the speed of the projectile equals the distance of its hodograph position to the origin. Therefore, depending on the initial position, the orbits fall into three cases:

1) They start at A or A', that is above the horizontal line $\dot{z} = -v_T$ and out of the semicircle (Fig. 1). In $v = -v_T\, sin\theta$ the first case the projectile is launched upwards whence in the second case is launched downwards. The speed will be decreasing until the minimum value $v_{\min}$ is achieved at M. Then it will be increasing converging to v_T (Fig. 3).

2) They start at B, inside or on the semicircle. Then speed is increasing for all time, converging to v_T. Minimum speed equals to v_0 (Fig. 1).

3) They start at C, that is below or on the horizontal line $\dot{z} = -v_T$. The speed is decreasing for all time, converging to v_T and there is no minimum speed (Fig. 1).

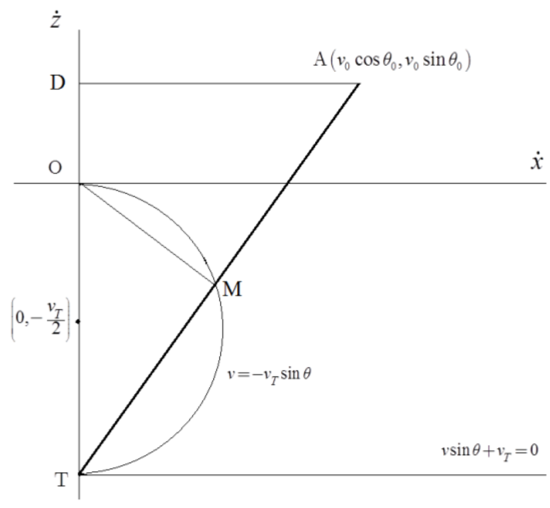

Figure 3. *The minimum speed* $v_{\min} = OM$ *occurs on the intersection of the hodograph AT and the semicircle*

The above remarks combine to the following.

Theorem 2: *A projectile launched in constant gravitational field under opposing resistance linear in speed will attain minimum speed,* $v_{min} = \dfrac{v_T\, v_0\, cos\theta_0}{\sqrt{v_0^2 + 2v_T\, v_0\, sin\theta_0 +\, v_T^2}}$ *exactly when the initial data* $(v_0\,, \theta_0)$ *satisfy the conditions* $v_0 + v_T \sin\theta_0 > 0$ *and* $v_T + v_0 \sin\theta_0 > 0$

Proof: The existence of the minimum exactly under the conditions, is clear by the aforementioned geometric analysis. We proceed to calculate *more geometrico* the value of the minimum speed. Let *D* be the orthogonal projection of *A* on the $\dot{z}$ axis (Fig. 3). Since *M* is on the semicircle, angle $\widehat{OMT} = \dfrac{\pi}{2}$. Consequently, triangles *TOM* and *TDA* are similar. Hence,

$$\frac{OM}{OT} = \frac{AD}{AT} \tag{18}$$

But $AD = v_0\, cos\theta_0$ and by the cosine law on triangle AOT

$$AT = \sqrt{AD^2 + 2AD\;DT\;cos\left(\frac{\pi}{2} + \theta_0\right) +\; DT^2} = \sqrt{v_0^2 + 2v_T\, v_0\, sin\theta_0 +\, v_T^2} \tag{19}$$

Combining (18) and (19) we get the value for $v_{\min}$

(20)

Let us further remark, since *OM* is perpendicular to *AT*, the corresponding tangent of the angle of θ_{min} inclination is given by

$$v_{min} = \frac{v_T\, v_0\, cos\theta_0}{\sqrt{v_0^2 + 2v_T\, v_0\, sin\theta_0 + v_T^2}} \tag{21}$$

CONCLUSIONS

We finish with some remarks about the educational merits of this presentation. It is the author's belief that dimensionless analysis, useful it may be at advanced level, hides from the student the physical aspects of the problem (Theorem 1 in our case) and thus, should be avoided. Physics and geometry stand a deep long relation. The advantage of the geometric method is that it gives the overall idea with one picture, providing a sense of elegance to the solution. Specific skills exercised by the students in the solution presented include polar coordinates in the velocity space, equations of straight line and circle in Cartesian and polar coordinates, distance, behavior of a function at critical points, 1st and 2nd derivative test from Calculus.

$$tan\theta_{min} = \frac{v_0 cos\theta_0}{v_0 sin\theta_0 + v_T}$$

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