Conical Pendulum – Linearization Analyses

Kevin Dean[1]
Jyothi Mathew[2]

Physics Department
The Petroleum Institute
Abu Dhabi, PO Box 2533
United Arab Emirates
[1]kdean@pi.ac.ae
[2]jmathew@pi.ac.ae

(Received: 08.02.2017, Accepted: 13.02.2017)

Abstract
A theoretical analysis is presented, showing the derivations of seven different linearization equations for the conical pendulum period T, as a function of radial and angular parameters. Experimental data obtained over a large range of fixed conical pendulum lengths (0.435 m – 2.130 m) are plotted with the theoretical lines and demonstrate excellent agreement. Two of the seven derived linearization equations were considered to be especially useful in terms of student understanding and relative mathematical simplicity. These linear analysis methods consistently gave an agreement of approximately 1.5% between the theoretical and experimental values for g, the acceleration due to gravity. An equation is derived theoretically (from two different starting equations), showing that the conical pendulum length L appropriate for a second pendulum can only occur within a defined limit: $L \geq [\, g / (4\, \pi^2)]$. It is therefore possible to calculate the appropriate circular radius R or apex angle ($0 \leq \phi \leq \pi / 2$) for any length L in the calculated limit, so that the conical pendulum will have a one second period. A general equation is also derived for the period T, for periods other than one second.
Keywords: Conical pendulum, theoretical linearization, experimental results

INTRODUCTION

The physics of an oscillating pendulum (simple and conical) can often be considered as somewhat challenging for pre-university students (Czudková and Musilová, 2000), but should not pose a significant problem for typical undergraduates. Conical pendulum lengths investigated experimentally (and published) usually range from approximately 20 cm (Tongaonkar and Khadse, 2011) up to greater than 3 m (Mazza, Metcalf, Cinson and Lynch, 2007).

The typical conical pendulum is a comparatively standard experiment that is frequently included in undergraduate laboratory syllabi. In some cases, the aim of the experiment is for undergraduate students to have the opportunity to design and then perform an experiment (either individually, or in a small team of up to three students). It is usual for students to record appropriate data in tabular form using suitable readily available software (such as Excel). The conical pendulum is also used to teach energy and angular momentum conservation (Bambill, Benoto and Garda, 2004) and potential energy (Dupré and Janssen, 1999) as well as the rotational dynamic interactions of classical mechanical systems (Lacunza, 2015).

Although the present work restricts the conical motion to being in a horizontal plane, it is possible to extend the mathematical analysis to three dimensions (Barenboim and Oteo, 2013). In the case of horizontal planar motion, this is frequently observed to be elliptical rather than truly circular, which has been studied in detail to determine the nature of orbital precession (Deakin, 2012). String tension measured as a function of the period (or rotational speed) has been documented, and is easily understood by pre-university (or first year university) students (Moses and Adolphi, 1998).

Based upon several years of experience acquired at The Petroleum Institute, the standard conical pendulum experiment is relatively straightforward for students to understand, then set up and obtain acceptable data. The principal aim of the conical pendulum as a design experiment is for students to investigate how the conical pendulum period T could depend on the radius R of the conical pendulum path, or on a measured angle as the physical variable. The experimental data analysis can be used to determine a value for g the acceleration due to gravity, (if the length of the conical pendulum is known), or the length L can be calculated assuming a value for the acceleration due to gravity.

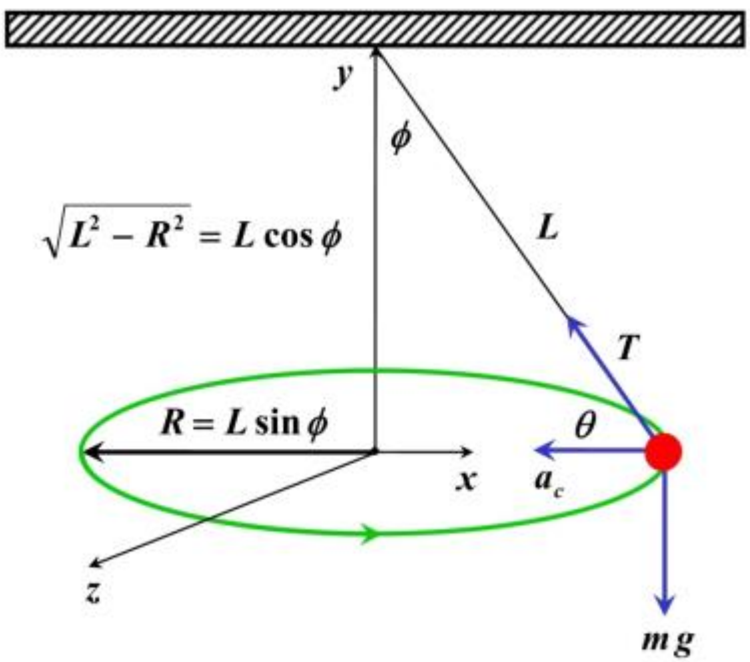

***Figure 1**. Conical Pendulum Schematic*

The pendulum mass (frequently referred to as a "bob") performs uniform horizontal circular motion over a range of circular radii such that $0 < R < L$ (where L represents the length of the pendulum string). It should be noted that the string is considered to be both mass-less and inextensible as well as being effectively devoid of air resistance (as is the pendulum mass itself).

The present purpose is to extend the effectiveness of this experiment by enabling students to produce suitable linear data, using one of the linearization methods that are derived in the subsequent theoretical section. Excel can then be used to prepare a straight-line chart, including a linear regression analysis, from which the slope (and) or the intercept can be used to obtain an unknown physical parameter (g or L). For the analysis that follows, it is not necessary to restrict the orbital period to approximate isochronous behavior, which would be the standard approach for analyzing the planar oscillations of a simple pendulum.

The analysis presented below applies to the full range of radii (subject to the physical restrictions that $R \le L$ and $\phi \le (\pi / 2)$, even though mathematically, the analysis allows for these unphysical limits). The physical implications of $R = L$ and $\phi = (\pi / 2)$ will be considered at the appropriate point(s) later in the analysis.

THEORETICAL ANALYSIS

The seven theoretical linearization analyses presented in this section are all based on the conical pendulum figure that is shown in the section above, which defines the appropriate physical parameters that will be used for each of the analyses:

DERIVATION OF THE CONICAL PENDULUM PERIOD *T*

From the figure, the period T can be readily derived for a conical pendulum of length L and with apex angle ϕ as shown above, where the subscript c indicates the centripetal acceleration and centripetal force acting on the conical pendulum mass:

$$F_c = m\,g\,\tan\phi = m\,a_c = \frac{m\,v^2}{R} \quad \Rightarrow v^2 = R\,g\,\tan\phi = \frac{g\,R^2}{\sqrt{L^2 - R^2}}$$

A straightforward re-arrangement of the above gives:

$$\frac{1}{v} = \sqrt{\frac{\sqrt{L^2 - R^2}}{g\,R^2}} \quad \Rightarrow \quad T = \frac{2\,\pi\,R}{v} = 2\,\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}}$$

In the form shown by the equation derived above for the period T, the conical pendulum period can be seen to be a definite function of the circular radius R although the functional relationship is neither straightforward nor linear. This functional relationship is illustrated in Chart 1 (Appendix 1), for nine different theoretical values of L along with the appropriate experimental data. The lengths of the conical pendulum for the experiments were in the range 0.435 m – 2.130 m (the theoretical and experimental data for specific lengths are indicated in the chart legend). Data for L = 0.199 m is included and referenced (Tongaonkar and Khadse, 2011).

For details of all the charts and the chart legend (which applies for all charts) refer to the Appendices.

Inspection of Chart 1 (Appendix 1) and the equation for the period, shows that the horizontal axis intercept (for $T = 0$) occurs for when $R = L$, whereas the vertical axis intercept occurs when the period T is the same as for a simple pendulum of the same length (with motion confined to the x–y or z–y plane for example, see Figure 1).

It is possible (and considered to be analytically desirable) to re-express the functional relationship between the conical pendulum period T and either the circular radius R, or the apex angle ϕ (or alternatively the angle θ measured with respect to the horizontal) to achieve linearization. Several straightforward methods of accomplishing the mathematical linearization are presented below and the most appropriate method(s) for ease of student use is specified in a later part of this analysis. Each of the charts that are shown contains both the theoretical calculations and experimental data values.

Linearization – Method 1

The starting point for this analysis is the period equation that was derived above, which is then squared as shown.

$$T = 2\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}} \quad \Rightarrow \quad T^2 = \left[\frac{4\pi^2}{g}\right]\sqrt{L^2 - R^2}$$

Chart 2 (Appendix 1) demonstrates this very straightforward method of linearization that directly produces a straight line passing through the origin of coordinates, which then enables the acceleration due to gravity g to be easily calculated from the slope. The mathematical analysis is uncomplicated and should be readily understood by typical students. By using a standard linear regression with the condition that the straight line should pass through the origin, this linearization method should produce good results.

Linearization – Method 2

The period equation that was derived earlier can be developed further to give,

$$T^4 = \left[\frac{16\pi^4 L^2}{g^2}\right] - \left[\frac{16\pi^4}{g^2}\right]R^2 \quad \Rightarrow \quad T^4 = c - m\,R^2$$

Although this analysis gives rise to an equation that involves the fourth power of the period T as a function of the conical radius squared, the equation is basically nothing more than a straight line, with a positive intercept that depends on the length L of the conical pendulum, and a negative slope that is independent of L. This can be clearly seen in the Chart 3 (Appendix 1), which shows the theoretical parallel straight lines and the experimental data that are displayed in Chart 1 (Appendix 1).

It can be noted that as a student exercise, it is possible (in principle) to calculate a value for L the length of a conical pendulum if this is not known in advance, when a value for the acceleration due to gravity g is provided. This is readily determined by calculating the numerical ratio of the vertical axis intercept divided by the slope, which gives L^2 and therefore L. Therefore, various values of L can be used by different student groups.

By referring to the linearization equation, it can be observed that the point where the line intercepts the horizontal axis corresponds to the situation where $R^2 = L^2$ so that L could be obtained directly.

Alternatively, if the pendulum length L is given, then both the slope and intercept can be used to calculate separate values for the acceleration due to gravity g.

Linearization – Method 3

The first analysis can be extended and re-arranged to make use of the fact that the straight line intercept $c = (mL^2)$ in the following way:

$$T^4 = m\left(L^2 - R^2\right) = \left[\frac{16\pi^4}{g^2}\right]\left(L^2 - R^2\right)$$

In this form, the linearization gives rise to a straight line with positive slope m passing through the origin of coordinates (which corresponds to the mathematical condition that $T = 0$ when $R^2 = L^2$). Although in practice this situation is clearly physically unattainable since it would require that the pendulum speed $v = \infty$, it is nevertheless mathematically precise and could be pointed out to students.

Inspection of the above linearization equation shows that the single straight line passes through the coordinate origin and has a constant positive slope that is independent of the pendulum length L. Consequently, therefore, all experimental data points (for all lengths of conical pendulum) will fit on the same straight line.

It is of interest to note that experimentally, having a large value for the conical pendulum length L would be highly beneficial in terms of obtaining good results compared to using a smaller length pendulum. This linearization is shown in Chart 4 (Appendix 1).

By knowing the conical pendulum length L and using the fact that theoretically the straight line must pass through the coordinate origin, it is only necessary for students to use suitable linear regression analysis, with the requirement that the trend-line line passes through the origin. It is then a simple matter for students to calculate a value for g (the local acceleration due to gravity) from the slope.

Linearization – Method 4

Starting with the linearization equation derived above, further analysis gives:

$$T^4 = \left[\frac{16\pi^4}{g^2}\right]\left(L^2 - R^2\right) \quad \Rightarrow \quad \ln\left(T\right) = \frac{1}{4}\ln\left[\frac{16\pi^4}{g^2}\right] + \frac{1}{4}\ln\left(L^2 - R^2\right)$$

This is calculated and shown in Chart 5 (Appendix 1).

All experimental data points must lie on a single straight line of constant positive slope having a value of exactly m = ¼ and with a vertical axis intercept that is independent of the conical pendulum length L. The intercept of the straight line on the vertical axis can be used to provide a value for g the local acceleration due to gravity. It can be seen that where the straight line crosses the horizontal axis, requires ln(T) = 0 (which therefore corresponds to the conical pendulum having a period T = 1 second). The physical consequences of this are outlined below.

$$\ln(T) = \frac{1}{4}\ln\left[\frac{16\,\pi^4}{g^2}\right] + \frac{1}{4}\ln\left(L^2 - R^2\right) = 0 \quad \Rightarrow$$

$$R^2 = L^2 - \frac{g^2}{16\,\pi^4} = L^2 \sin^2\phi \quad \Rightarrow \quad L^2\left(1 - \sin^2\phi\right) = \frac{g^2}{16\,\pi^4} \quad \Rightarrow$$

$$L^2 \cos^2\phi = \frac{g^2}{16\,\pi^4} \quad \Rightarrow \quad L\cos\phi = \frac{g}{4\,\pi^2} \quad \Rightarrow \quad \cos\phi = \frac{g}{4\,\pi^2\,L} \quad \Rightarrow$$

$$0 \le \frac{g}{4\,\pi^2\,L} \le 1 \quad \Rightarrow \quad L \ge \frac{g}{4\,\pi^2}$$

Inspection of the final equations in the above analysis shows that the conical pendulum length L that is appropriate for a second's pendulum, falls within a precisely defined range. This range limit for L will re-occur at the end of a subsequent derivation in a later section, and will be considered in context with the simple pendulum.

This linearization analysis is considered to be somewhat complicated for use by most students taking a first level Physics course. As a result, the application of logarithmic functions by students to experimental data analysis and interpretation is avoided whenever possible. This is due to an intrinsic lack of in-depth understanding of the logarithmic function and how these should be treated.

Linearization – Methods 5 & 6

By making reference to the conical pendulum figure, it can be noted that the following two trigonometric functions can be obtained:

$$\sin\phi = \frac{R}{L} \quad \Rightarrow \quad R = L\sin\phi \qquad \textit{and} \qquad \cos\theta = \frac{R}{L} \quad \Rightarrow \quad R = L\cos\theta$$

It now becomes a simple matter to substitute for R to give the following:

$$R^2 = (L \sin\phi)^2 = L^2 \sin^2\phi \quad \Rightarrow$$

$$T^4 = \left[\frac{16\,\pi^4\,L^2}{g^2}\right] - \left[\frac{16\,\pi^4\,L^2}{g^2}\right]\sin^2\phi$$

This is once again in the same format as the first linearization analysis where the intercept *c* equals the slope *m* and gives rise to the following two equations:

$$T^4 = \left[\frac{16\,\pi^4\,L^2}{g^2}\right](1 - \sin^2\phi) = \left[\frac{16\,\pi^4\,L^2}{g^2}\right]\cos^2\phi$$

The first linearization equation is shown in Chart 6 (Appendix 1), which again includes the theoretical straight lines and the experimental data sets that were shown in Chart 1 (Appendix 1) earlier: It is interesting to make note of the fact that the horizontal axis intercept occurs at exactly 1 is a direct result of the fact that $T = 0$ when $\phi = \pi/2$ rad (or 90°). This mathematical result can be thought of as defining an "anchor" point on the horizontal axis, for the straight-line data chart.

An alternative substitution for the conical pendulum radius *R* (using the second equation for the cosine function in place of the sine) will give rise to the linearization shown in Chart 7 (Appendix 1). For this representation, the origin of coordinates is obtained as a direct result of the fact that $T = 0$ when $\phi = (\pi / 2)$ rad (or 90°). As a result of the straight line passing through the coordinate origin, this analysis is slightly more preferable for student use, when compared to the linearization of Chart 6 (Appendix 1). When experimental data is plotted as a chart, it can be used to obtain a value for either *g* if *L* is given, or *L* if *g* is provided.

Linearization – Method 7

It is possible to extend the second version of the two previous linearization equations by taking natural logarithms.

$$T^4 = \left[\frac{16\,\pi^4\,L^2}{g^2}\right]\cos^2\phi \quad \Rightarrow \quad 4\ln(T) = \ln\left[\frac{16\,\pi^4\,L^2}{g^2}\right] + 2\ln(\cos\phi)$$

$$\ln(T) = \frac{1}{4}\ln\left[\frac{16\,\pi^4\,L^2}{g^2}\right] + \frac{1}{2}\ln(\cos\phi) = \frac{1}{4}\ln\left[\frac{16\,\pi^4\,L^2}{g^2}\right] + \frac{1}{2}\ln(\cos\phi)$$

Using the same data sets as for earlier analyses produces Chart 8 (Appendix 1).

This linearization gives rise to a series of parallel straight lines with slope $m = ½$ and with a vertical axis intercept that will provide a value for either g if L is given, or L if g is known. The analysis below uses an abbreviated equation form.

$$\ln(T) = \frac{1}{4}\ln(a) + \frac{1}{2}\ln(\cos\phi) = 0 \quad \Rightarrow \quad \frac{1}{2}\ln(a) + \ln(\cos\phi) = 0$$

$$\ln\left(\sqrt{a}\right) + \ln(\cos\phi) = 0 \quad \Rightarrow \quad \ln(\cos\phi) = -\ln\left(\sqrt{a}\right) = \ln\left(\frac{1}{\sqrt{a}}\right)$$

$$\cos\phi = \frac{1}{\sqrt{a}} = \sqrt{\frac{g^2}{16\pi^4 L^2}} \quad \Rightarrow \quad \cos\phi = \frac{g}{4\pi^2 L}$$

Inspection of the final equation above shows (again) that the conical pendulum length that is appropriate for a second's pendulum falls within a well-defined range. The range of acceptable values for the cosine function readily gives the simple analysis below.

$$0 \le \frac{g}{4\pi^2 L} \le 1 \quad \Rightarrow \quad L \ge \frac{g}{4\pi^2}$$

It is therefore possible to determine the appropriate apex angle for any length within the calculated range, such that the conical pendulum will have a one second period. It is of interest to note that in terms of the linearization of a simple pendulum (see later section for full details), the above limit equation can be expressed below, where the simple pendulum slope is given by m_{SP} (see later).

$$L \ge \frac{g}{4\pi^2} \quad \Rightarrow \quad L \ge \frac{1}{m_{SP}}$$

Using the local value for acceleration due to gravity $g = 9.79\ \mathrm{ms}^{-2}$ (2 decimal places) gives, $L > 0.24798$ m (24.798 cm). The standard value for the acceleration due to gravity $g = 9.81\ \mathrm{ms}^{-2}$ (2 decimal places) gives, $L > 0.24849$ m (24.849 cm).

This is the same as the small-angle approximation period T for the simple pendulum undergoing simple harmonic motion, as shown below setting $T = 1$:

$$T = 2\pi\sqrt{\frac{L}{g}} = 1 \quad \Rightarrow \quad L = \frac{g}{4\pi^2}$$

Reference to Chart 1 (Appendix 1) confirms that a horizontal line at T = 1 intercepts all the lines for pendulum lengths satisfying the above analysis. If periods other than one second are to be specifically investigated, then a simple re-arrangement of the initial period equation will enable such a study, by using:

$$T = 2\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}} \quad \Rightarrow \quad R(T) = \frac{\sqrt{16\pi^4 L^2 - g^2 T^4}}{4\pi^2}$$

Additional details regarding the functional dependence of the conical pendulum length on rotational angular speed and the derivation of the critical length can be obtained from published previously published work (Klostergaard, 1976).

LINEARIZATION OF A SIMPLE AND CONICAL PENDULUM COMPARED

Squaring both sides of the familiar SHM pendulum equation readily gives the familiar straight line equation, with the square of the pendulum period T^2 being directly proportional to the pendulum length L. For the purpose of comparison with the present work, the nearest equivalent linearization for the conical pendulum is also shown. In the linearization equations, the subscripts SP and CP refer to the simple pendulum and conical pendulum respectively.

It can be observed that these two pendulums exhibit similar physical behavior, which is inextricably linked through the mathematical term ($4\pi^2/g$) that appears as the slope m_{SP} of the straight line for the simple pendulum, and appears squared as the slope m_{CP} of several of the conical pendulum linearization equations.

$$T_{SP}^2 = \underbrace{\left[\frac{4\pi^2}{g}\right]}_{\text{Slope } m_{SP}} L \quad \Rightarrow \quad T_{SP}^4 = m_{SP}^2\, L^2$$

$$T_{CP}^4 = \underbrace{\left[\frac{16\pi^4}{g^2}\right]}_{\text{Slope } m_{CP}} \left(L^2 - R^2\right) = m_{SP}^2 \left(L^2 - R^2\right)$$

Due to the absence of a vertical axis intercept, both T^4 equations provide a simple and direct technique of displaying experimental data that yields a straight line passing through the coordinate origin. It can be seen that the slope for both pendula is the same, being the square of the slope for a simple pendulum.

In terms of experimental usefulness, a single calculation of the slope of the best straight line passing through the origin will give g the local value for the acceleration due to gravity. It is noted that the conical pendulum length L remains constant, but not for the simple pendulum.

CONICAL PENDULUM EXPERIMENT AND DATA

When a conical pendulum experiment is performed, there are several parameters that can be quantified and measured. It is usual to have some form of timing measurement to determine the orbital period T and some way of measuring any one (or several) of the following: the orbital radius R, the cone apex angle ϕ or equivalently, the string angle θ measured with respect to the horizontal.

***Table 1**. Experimental Data (*Tongaonkar and Khadse, 2011)*

L (m) = 2.130			L (m) = 2.003			L (m) = 1.875		
R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)
0.200	5.3878	2.940	0.200	5.7305	2.854	0.200	6.1232	2.748
0.250	6.7404	2.930	0.300	8.6139	2.837	0.300	9.2069	2.736
0.300	8.0967	2.925	0.400	11.5194	2.829	0.400	12.3178	2.730
0.400	10.8240	2.905	0.500	14.4554	2.823	0.500	15.4660	2.716
0.500	13.5764	2.890						
0.600	16.3611	2.875						
L (m) = 1.723			L (m) = 1.485			L (m) = 1.252		
R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)
0.200	6.6657	2.641	0.200	7.7401	2.440	0.200	9.1921	2.246
0.300	10.0272	2.628	0.300	11.6551	2.441	0.300	13.8639	2.235
0.400	13.4239	2.617	0.400	15.6262	2.423	0.400	18.6320	2.203
0.500	16.8694	2.603	0.500	19.6760	2.393	0.500	23.5382	2.176
L (m) = 1.029			L (m) = 0.825			L (m) = 0.435		
R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)	R (m)	*Apex* (deg)	T (s)
0.200	11.2075	2.028	0.200	14.0297	1.806	0.050	6.6003	1.327
0.300	16.9505	2.002	0.300	21.3237	1.798	0.100	13.2903	1.312
0.400	22.8753	1.986	0.400	29.0025	1.732	0.200	27.3723	1.286
0.500	29.0720	1.933						
			*EJPE Data L (m) = 0.199					
			R (m)	*Apex* (deg)	T (s)			
			0.0681	19.9990	0.850			
			0.1197	36.9658	0.775			
			0.1735	60.6627	0.625			
			0.1924	75.1505	0.460			
			0.1949	78.4043	0.400			

The experimental set-up required two slightly different methods of rigid support depending on the pendulum lengths *L* to be investigated. Initially, a light-weight inextensible string of length 1.500 m is held between a pair of parallel-sided wooden blocks, which are tightened together using clamps. A concentrically scaled pendulum orbit sheet is placed directly below the clamp. Then a spherical steel bob is then connected so that it is just above the center of the orbit sheet when it is at a vertical stationary equilibrium. The length of the pendulum is measured from the bottom edge of the clamping blocks to the center of the bob.

A force is applied on the bob at a desired radius *R* directly along its tangent. For a single measurement at radius *R*, a digital stopwatch is used to record the time for 5 rotational periods (referred to as "orbits" in the experiment). To ensure consistent accurate measurements are obtained, this procedure is repeated at least 5 times. The radius of the conical pendulum orbits is then changed in order to perform the experiment as a function of radius.

This procedure is then repeated for different radii and lengths. When sufficient good quality experimental data is obtained, the periods and other parameters are calculated and the experimental data is plotted on the same chart as the theoretical lines. The two methods of pendulum support (for both a long and short conical pendulum) are shown photographically below the data table.

The table below shows typical experimental data for conical pendulum lengths in the range $0.435 \text{ m} \leq L \leq 2.130 \text{ m}$ that were obtained by the authors. Data previously obtained by other authors (Tongaonkar and Khadse, 2011) and published is also indicated in the table below ((EJPE 0.199 m).

For pendulum string lengths, more than 1.500 m, the string support clamp is attached directly to a rigidly fixed ceiling rod, which maintains a constant (but adjustable) conical pendulum length *L*.

Figure 2. *Short pendulum L*

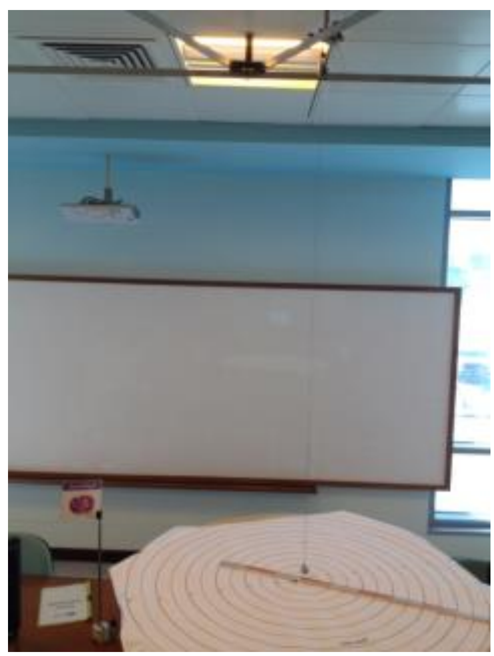

Figure 3. *Long pendulum L*

DISCUSSION OF RESULTS AND CONCLUSIONS

Seven theoretical methods of linearization have been derived for the conical pendulum period and all have been plotted (using Excel) on appropriate charts for the length range 0.435 m ≤ L ≤ 2.130 m (with additional calculations for L = 0.199 m). The charts demonstrate excellent agreement between the theoretical analysis and experimental data over the length range considered. Theoretical calculations used the local value of the gravitational acceleration g = 9.79 m s^{-2}.

Although all the seven linearization equations are equally capable of providing very closely similar values for the local gravitational acceleration, not all the derived linearization equations are equally simple to apply. As a result, the equations are not necessarily all suitable for use by students.

Inspection of the seven theoretically derived linearization equations as well as their appropriate charts, suggests that those charts having a straight line passing through the coordinate origin would be most appropriate for student use.

The table below summarizes the experimental results obtained from the three linearization charts (Charts 2, 4 and 7) that were considered to be most suitable for undergraduate student experimental classes (Appendix 1). In the table of results shown below, the following is defined: $\Delta g = |(\text{Local value}) - (\text{Experimental value})|$.

***Table 2**: Summary of Results (*Tongaonkar and Khadse, 2011)*

Theory L (m) & Expt L (m)	Chart 2		Chart 4		Chart 7	
	Slope m	Calculated g	Slope m	Calculated g	Slope m	Calculated g
2.130	4.05	9.746	16.41	9.745	74.46	9.745
2.003	4.08	9.666	16.68	9.666	66.92	9.666
1.875	4.06	9.721	16.49	9.722	57.97	9.722
1.723	4.09	9.664	16.69	9.665	49.54	9.665
1.485	4.09	9.664	16.69	9.665	36.80	9.665
1.252	4.10	9.628	16.81	9.629	26.35	9.629
1.029	4.11	9.601	16.88	9.608	17.88	9.608
0.825	4.14	9.527	17.15	9.532	11.67	9.532
0.435	4.13	9.558	17.00	9.575	3.22	9.575
:	Average g =	9.642	Average g =	9.645	Average g =	9.645
:	Δg =	0.148	Δg =	0.145	Δg =	0.145
:	Δg =	1.5%	Δg =	1.5%	Δg =	1.5%
*0.199	3.87	10.213	14.77	10.271	0.59	10.271

In each of the above three charts, the calculated straight lines were constrained to pass through the origin of coordinates (which theory shows to be a necessary physical condition). It can be seen that the theoretical derivations are completely self-consistent and give rise to uniform values for the gravitational acceleration, using the experimental data that is presented.

This analysis method consistently provided an agreement of approximately 1.5% between theoretical and experimental values for *g*, the acceleration due to gravity (the local value being $g = 9.79\ m/s^2$). Previously published results (Mazza, Metcalf, Cinson and Lynch, 2007) for conical pendulum experiments having a length range that extended up to approximately 3 m (actual range: $1.192\ m \leq L \leq 3.411\ m$) were reported to be usually better than 2%.

In conclusion, it can therefore be stated that the theoretical derivations have been thoroughly tested and confirmed as correct for the range of conical pendulum lengths investigated, including previously published results (Tongaonkar and Khadse, 2011) for small *L* and (Mazza, Metcalf, Cinson and Lynch, 2007) for large *L*.

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the editor EJPE for granting permission to include previously published experimental data from Tongaonkar and Khadse (2011), for a conical pendulum of length 0.199 m, which is referred to in the text as (EJPE 0.199 m).

APPENDIX 1. CHARTS FOR THEORETICAL & EXPERIMENTAL RESULTS

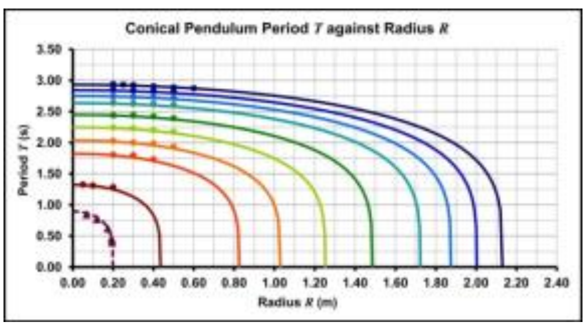

Chart 1 – Equation 1

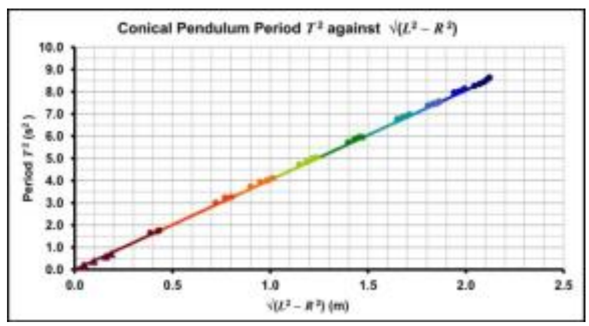

Chart 2 – Equation 2

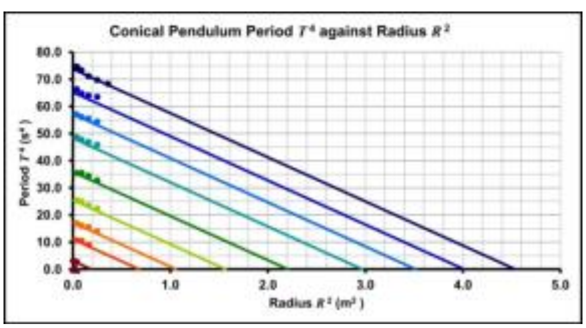

Chart 3 – Equation 3

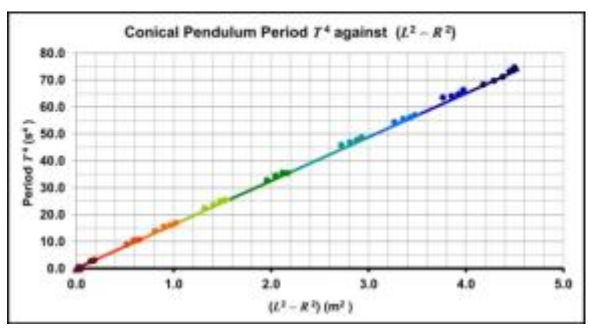

Chart 4 – Equation 4

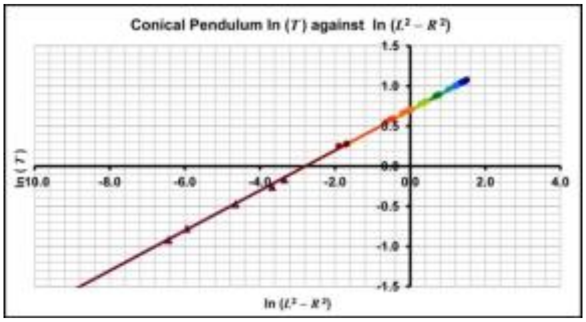

Chart 5 – Equation 5

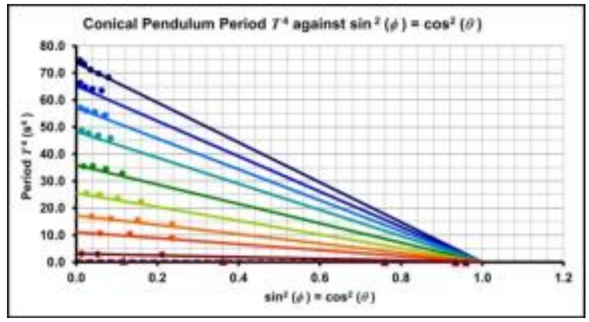

Chart 6 – Equation 6

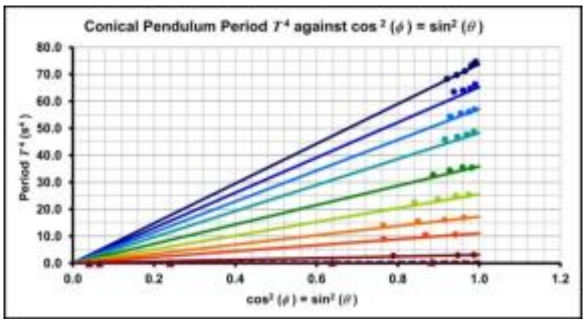

Chart 7 – Equation 7

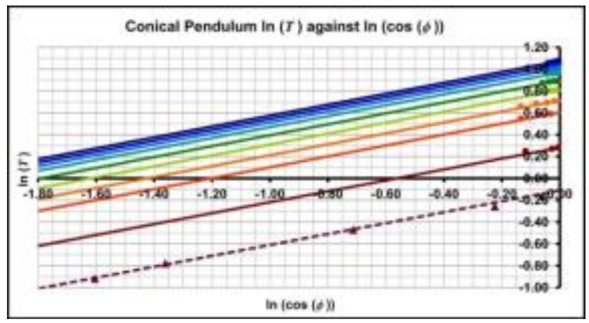

Chart 8 – Equation 8

APPENDIX 2 – CHART LEGEND FOR THEORETICAL & EXPERIMENTAL RESULTS

Theoretical and experimental results are indicated from largest conical pendulum length (L = 2.130 m) to the shortest (L = 0.435 m). Calculations for L = 0.199 m are also included, with experimental data that was obtained by other authors (Tongaonkar and Khadse, 2011) and has been previously published (referred to as EJPE 0.199 m).

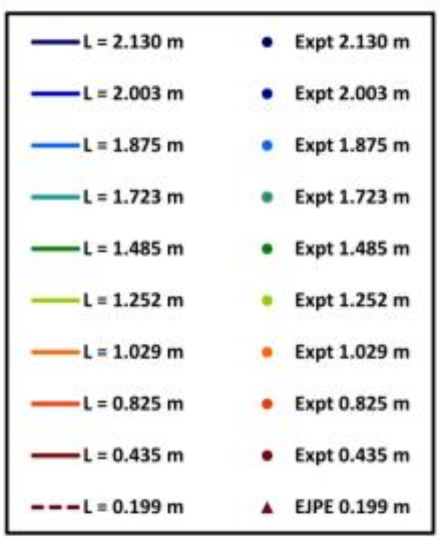

Figure 2 - Chart Legend

APPENDIX 3 – EQUATIONS USED FOR THEORETICAL CALCULATIONS

$T = 2\pi\sqrt{\frac{\sqrt{L^2 - R^2}}{g}}$ Equation 1 – Chart 1	$T^2 = \left[\frac{4\pi^2}{g}\right]\sqrt{L^2 - R^2}$ Equation 2 – Chart 2
$T^4 = \left[\frac{16\pi^4 L^2}{g^2}\right] - \left[\frac{16\pi^4}{g^2}\right]R^2$ Equation 3 – Chart 3	$T^4 = \left[\frac{16\pi^4}{g^2}\right]\left(L^2 - R^2\right)$ Equation 4 – Chart 4
$\ln(T) = \frac{1}{4}\ln\left[\frac{16\pi^4}{g^2}\right] + \frac{1}{4}\ln\left(L^2 - R^2\right)$ Equation 5 – Chart 5	$T^4 = \left[\frac{16\pi^4 L^2}{g^2}\right] - \left[\frac{16\pi^4 L^2}{g^2}\right]\sin^2\phi$ Equation 6 – Chart 6
$T^4 = \left[\frac{16\pi^4 L^2}{g^2}\right]\cos^2\phi$ Equation 7 – Chart 7	$\ln(T) = \frac{1}{4}\ln\left[\frac{16\pi^4 L^2}{g^2}\right] + \frac{1}{2}\ln(\cos\phi)$ Equation 8 – Chart 8

REFERENCES

Bambill, H.R., Benoto, M.R. and Garda, G.R., (2004), *European Journal of Physics, Vol 25, p31-35*

Barenboim, G. and Oteo, J.A., (2013), *European Journal of Physics, Vol 34, p1049-1065*

Czudková, L. and Musilová, J., (2000), *Physics Education, Vol. 35, Number 6, p428-435*

Deakin, M.A.B., (2012), *International Journal of Mathematical Education in Science and Technology, Vol. 44, Issue 5, p745-752, 2013*

Dupré, A. and Janssen, P., (1999), *American Association of Physics Teachers, Vol. 68, p704-711*

Klostergaard, H., (1976), *American Association of Physics Teachers, Vol. 44, p68-69*

Lacunza, J.C., (2015), *Journal of Applied Mathematics and Physics, Vol. 3, p1186-1198*

Mazza, A.P., Metcalf, W.E., Cinson, A.D. and Lynch, J.J., (2007), *Physics Education, Vol. 42, Number 1, p62-67*

Moses, T. and Adolphi, N.L., (1998), *American Association of Physics Teachers, Vol. 36, p372-373*

Tongaonkar, S.S. and Khadse, V.R. (2011), *European J of Physics Education, Vol. 2, Issue 1, 2011*