EJPE european journal of physics education

European J of Physics Education Volume 8 Issue 2 1309-7202 Arlego & Fanaro

# Light and Matter Diffraction from the Unified Viewpoint of Feynman's Sum of All Paths

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(Received: 05.04.2018, Accepted: 22.05.2017)

**Abstract**
In this work, we present a pedagogical strategy to describe the diffraction phenomenon based on a didactic adaptation of the Feynman's path integrals method, which uses only high school mathematics. The advantage of our approach is that it allows to describe the diffraction in a fully quantum context, where superposition and probabilistic aspects emerge naturally. Our method is based on a time-independent formulation, which allows modelling the phenomenon in geometric terms and trajectories in real space, which is an advantage from the didactic point of view. A distinctive aspect of our work is the description of the series of transformations and didactic transpositions of the fundamental equations that give rise to a common quantum framework for light and matter. This is something that is usually masked by the common use, and that to our knowledge has not been emphasized enough in a unified way. Finally, the role of the superposition of non-classical paths and their didactic potential are briefly mentioned.
**Keywords**: quantum mechanics, light and matter diffraction, Feynman's Sum of all Paths, high education

## INTRODUCTION

This work promotes the teaching of quantum mechanics at the basic level of secondary school, where the students have not the necessary mathematics to deal with canonical models that uses Schrodinger equation. The alternative teaching line of quantum mechanics that uses Feynman´s approach started in 1998 (Taylor et al. 1998), based on the nontechnical Feynman's book (Feynman, 1985) 'QED: The Strange Story of Light and Matter'. Although the path integral technique is a sophisticated tool, indispensable in advanced areas of physics such as quantum field theory, Feynman showed that at least the essence of the method is revealed through a simple mathematics, consisting of adding 'arrows'.

The early works triggered a considerable amount of teaching proposals based on the transposition of the path integrals method. In this context, transposition refers to the idea of "didactic transposition" (Chevallard, 1985), i.e. the meaningful transformation of a highly technical knowledge into an accessible one, without losing its nature. This has led to the recognition that the Feynman approach and some of its derivations serve as an interesting introduction to the foundations of quantum mechanics to the students. The strength of the method consists in that it allows to describe basic aspects of quantum mechanics associated with light and matter, in a unified way and with elementary mathematical means, as we will illustrate in this work. Several works have addressed the Feynman method (Hanc et al. 2003; Styer, 2000; Dobson et al. 2006; Ogborn and Taylor, 2005; Beau, 2012; Malgieri et al., 2014; Malgieri et al., 2015, 2017) and our contribution to the subject ranges from proposals of didactic sequences to the analysis of the implementations in secondary school courses (Fanaro et al., 2008, Fanaro et al., 2009; Fanaro et al., 2012; Fanaro et al., 2012A; Fanaro et al., 2012B; Fanaro et al., 2014).

The teaching of the diffraction phenomenon has been traditionally introduced from a classical wave viewpoint, where the textbook approach is based on the context of electromagnetic waves (Born and Wolf, 1999). Usually at the basic level (Halliday et al., 2011) certain limit cases such as the Frauenhofer diffraction are developed, where the solution does not use the Maxwell equations or the Kirchhoff's theory explicitly. Although the construction based on the Huygens-Fresnel principle is suitable, it is slightly difficult to introduce the subject even at the high school level (Temes, 2003).

The analysis of the problems and difficulties that the students face with the wave treatment of these phenomena, has been investigated in several works (Colin and Viennot, 2001; Ramil et al., 2007; Wosilait et al., 1999; Maurines, 2010) although this is not the focus of the present paper.

The quantum diffraction phenomenon can be analyzed from exact (in very exceptional cases), perturbative, semi classical and numerical viewpoints. The diffraction of electrons through a single finite slit has been considered from the point of view of wave mechanics, via the Schrodinger equation (Gitin, 2013; Michelini and Stefanel, 2008) and path integrals (Feynman and Hibbs, 1965), whereas the quantum theory of light diffraction is considered in Wua et al. (2010).

This paper is structured as follows. In section II, we show how to build a *time independent* quantum mechanical model for the diffraction using an adaptation of the Feynman's method. We obtain a general expression applied to the diffraction of a nonrelativistic particle of mass *m* (e.g. electrons) in section III, and subsequently to the case of the light, in section IV. Finally, in section V, we outline a didactic strategy to teach this issue at high school level.

## DIFFRACTION BY A SINGLE SLIT FROM FEYNMAN'S APPROACH

When an electron beam passes through a slit of adequate size, a special distribution appears on a detection screen behind the slit. This pattern contains a central maximum, where most electrons are detected. Around the central maximum, other lateral and successively smaller maxima appear, indicating that some electrons are also detected there. In addition, between these maxima there are places on the detection screen where there is virtually no electron detection. This experiment, imagined in the first times of the quantum mechanics could be carried out many years later (Tonomura et al., 1989; Bach et al., 2013) revealing one of the puzzling aspects of the behavior of the matter at atomic scale.

However, this unexpected feature from the everyday perspective of particle is totally natural and intuitive from a wave viewpoint. In fact, the interference of waves is a usual phenomenon that can give rise to an alternating pattern of maxima and minima as observed in this experiment. On the other hand, when the slit is enlarged the main maximum becomes more intense, indicating that practically all electrons are detected there, and the secondary ones tend to disappear. In this limiting case, the electrons behave more like a beam of particles, copying the profile of the slit on the screen.

The light behaves similarly: when monochromatic light (laser) from a distant source passes through a narrow slit and is detected on a screen, the light distributes in a characteristic pattern consisting of a broad and intense central maximum, surrounded by narrower and less intense maxima. Performing this experiment with special conditions of the light (for example with light of very weak intensity), we obtain a pattern that resembles and shares common characteristics with the electrons distribution. Moreover, when the slit is enlarged the light passes and reproduces the shape of the slit marking a defined border between light and shadow on the screen.

In this way, the properties of light and matter that exhibit a common phenomenology in many respects. This suggests a unified description in terms of a model that deals in the same terms with light and matter. The *quantum* model gives an appropriate via to this purpose and here we will consider a didactic transposition of the Feynman method.

First, let us stress that in quantum mechanics the certainty of a given event is not predicted, as it is in classical physics. In quantum mechanics, the event probability is predicted in relation to other events (relative probability). For example, the rules of quantum mechanics predict the probability that an electron or light emitted from the source is detected at a given point on the screen behind a finite slit.

The way in which the time independent probability of a given event is calculated in the adapted Feynman method presented here can be summarized in the following sequence of steps.

*(I)* *The initial (i) and final (f) state of the event are identified. In our case the initial state is the emission of monochromatic (light) / mono-energetic (matter) and the final state is the detection on the screen.*

*(II)* *For each real space path connecting (i) with (f) we associate a two-dimensional vector (a small arrow in the popular language of Feynman[2]. In the context of this work its length will always be the same for all paths and we will arbitrarily assign a unit length. On the other hand, its direction or phase will be fixed by the angle $\phi$ that it forms with the x-axis.*

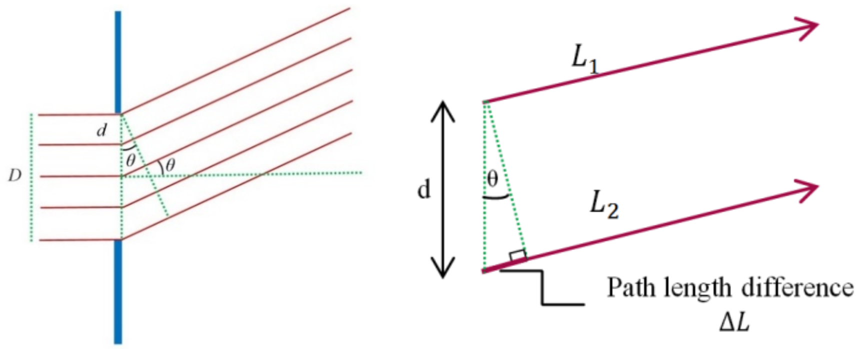
*(III)* *The angle associated with each path (phase) is proportional to the length (L) of the path, i.e. $\phi = C\,L$. The constant of proportionality C will depend on the case in question, matter or light, but it will have an inverse of length dimension. This association of the phase with the length path is not entirely general, but it serves our purpose. In this sense, it should be considered as part of the didactic adaptation, as we will discuss later.*

(IV) *Finally, we add all the vectors corresponding to the different paths connecting (i) with (f). The square module of the resulting vector is proportional to the probability of the event considered.*

In practical terms, the sum of all infinite contributions from the different paths represents a major difficulty in Feynman's original method, even more in a version adapted for didactic purposes like this one. To address this problem, we rely on the special role that the 'classical' paths can play, which in turn is reminiscent of well-established techniques of semi-classical approximation in quantum mechanics (Shankar, 1980). The central idea here is that in many circumstances the shortest paths -which in a broad sense are the paths followed by macroscopic objects- along with those of their close environment, have a dominant contribution to the sum. The reason is that for arbitrary paths their associated vectors will point to arbitrary directions because their lengths are not related. The collective effect of these majoritarian paths will be a mutual 'statistical' cancellation, since there is no preferred direction where to point. However, the vectors associated with the shortest path and its surroundings will point approximately in the same direction, that is, they will contribute in phase to the sum. This is because the variations around the minimum are smaller than around any other point (excepting the maximum not considered here), which is related with the basic property of the extremes of a function.

The strategy here will be to consider the family of shortest classical paths, and totally disregard the immense majority represented by the remaining contributions. This transforms the problem of the sum into something approachable from a didactic perspective. Although at first glance this would appear to strip away the quantum character of the model, as we shall see, it remains sufficient for the quantum aspects to survive, mainly through the way in which the different classical paths interfere in the calculation of probability (step IV).

To apply this strategy to our problem we will consider that both the source and the detection screen are infinitely far from the central screen containing the slit, i.e. the special case of Frauenhofer diffraction. Since the path differences occur only from the slit to the detection point, we only consider this part of parallel paths in direction to that point, as shown in Figure 1(a). For teaching purposes, we will consider only a subset of *N* equally spaced short paths along the slit, which are sufficient to capture the physics of the problem. The reason for choosing equally spaced paths is that the construction of the sum becomes very simple and allows us to do a direct geometric interpretation.




(a) (b)

***Figure 1****. (a) Schematic setup of the single slit diffraction, considering a set of N alternative paths. The source and the detection screen (not shown) are very far left and right respectively. (b)Detail of two consecutive paths and the path length difference which will play a relevant role in the discussion.*

To this aim, let us consider Figure 1 (b) where a detail of the paths is shown. As it can be seen, the difference in length $\Delta L$ between two successive paths is constant:

$$\Delta \mathrm{L} = \mathrm{d} \sin(\theta) \qquad (1)$$

This makes the phases associated with adjacent paths also vary in the same way, since the phases are proportional to the lengths of the paths. In terms of the distance $d$ between paths along the slit and the angle pointing in the direction of detection $\theta$, the phase variation $\Delta\phi$ between two neighboring paths is

$$\Delta\phi = C\,\Delta L = C\,d \sin\theta \qquad (2)$$

Figure 2 shows an example of the sum (step IV) of *N=7* paths, where the vectors form an open polygon of *N* sides and the vertices are on a circle. Although this sum can easily be done analytically, it is instructive from a didactic viewpoint to approach it geometrically.

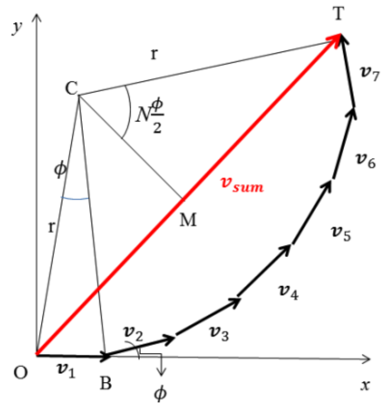



***Figure 2**. The sum of (N=7) direct classical paths with successive phase differences $\Delta\phi$. In this construction, for simplicity the phase of the first vector is arbitrarily considered zero, and therefore the successive phase differences are directly indicated as $\phi$. This notation will be used hereafter. Note that the vector angle from the x-axis sequentially increases by (n-1)*

By construction note that if $r$ is the radius of the circle in Figure 2, it satisfies on the one hand:

$$|\boldsymbol{v_m}| = 2\, r \sin\left(\frac{\phi}{2}\right)$$

On the other hand, for the resulting vector: $|v_{sum}| = 2\, r \sin(\frac{N\,\phi}{2})$. Combining these two results to eliminate $r$, and considering that $|\boldsymbol{v_m}| = 1$ we get:

$$|\boldsymbol{v_{sum}}| = \frac{\sin\left(\frac{N\phi}{2}\right)}{\sin\left(\frac{\phi}{2}\right)} \qquad (3)$$

which squared gives the relative probability,

$$p(\phi) \propto \frac{\sin^2\left(\frac{N\phi}{2}\right)}{\sin^2\left(\frac{\phi}{2}\right)} \qquad (4)$$

This expression gives the detection probability of light detection in a certain direction given by $\theta$ (Fig. 1a).

The behavior of the function (4) is determined by its quotient. For $\phi = 0$ the function tends to $N^2$ at that limit, since the sine and the argument are approximately equal here. On the other hand, $\phi = \frac{2}{N}\pi$ corresponds to the first minimum in the P($\phi$) curve. Analogously, the second maximum is determined, being its amplitude much smaller than the main maximum. With this analysis, we can describe the dependence of the probability distribution in terms of $\phi$. However, in pedagogical terms it is much more efficient to explore the geometric construction of the sum as shown in Figure 3.

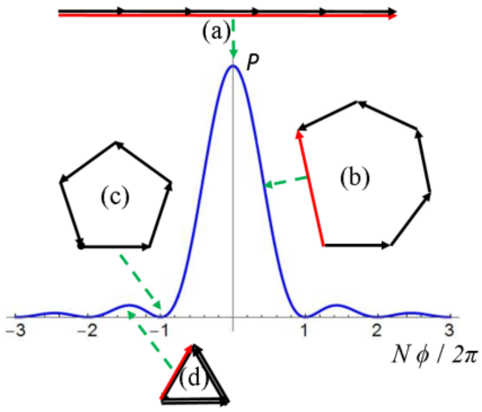



*Figure 3. General structure of quantum Frauenhofer diffraction. Blue solid line indicates the probability distribution (3) on the detection screen. The insets illustrate the construction of the distribution curve from the quantum superposition of N= 5 classical paths. For each phase difference $\phi$ the contributions of the paths (black arrows) form a regular polygonal chain. The corresponding probability is proportional to the square length of the resulting vector (red arrows). When the phase difference $\phi = 0$, as in case (a) all paths are equivalent (the chain is totally flat) and the contribution to the probability is maximal. By increasing $\phi$, the chain curves and the probability decreases as in case (b). Following this trend there would be a phase difference where the chain closes on itself, forming in this case a pentagon as shown in (c). In this case, the probability becomes zero. Finally, case (d) illustrates the situation where the phase difference is such that the chain completes more than one cycle without closing, giving a relative maximum to the probability. This structure of a central maximum surrounded by lobes of smaller maxima is repeated for $\phi \bmod 2\pi$ (not shown here).*

Here we consider *N=5* to better visualize the geometric figures formed. The central maximum is produced by the in-phase contribution of all the paths, and the minimum by the total cancellation through the whole number of turns. The intermediate regions are the product of partial cancellation. The caption of Figure 3 describes in more detail the structure of the polygonal formed in this case.

At this moment, the reader may have the impression that the well-known phasor method, commonly used in optics and, in general, in any wave problem (addition of electromagnetic fields, composition of harmonic motions, etc.) is simply being described to determine the amplitude from a series of partial components (close circles for nodes, in diffraction, and open arcs for any other case). However, the novelty that presents this work is that the Integral Path method (in its adapted version for didactic proposes) uses the mathematics of phasors in the context of quantum mechanics and analyze its conceptual consequence: it allows to analyze with students that do not have sophisticated mathematics tools, the phenomenon of diffraction, from a model that overcomes the wave-matter opposition.

So far, the discussion is very general and does not make specific allusion to electrons or light so it can form the basis of an approach that can extend in several directions. Some of the different possibilities are explored in the next sections.

## ELECTRONS DIFFRACTION

In our proposal for electrons (and matter in general) we will consider situations independent of time, where the energy $E$ is conserved. By avoiding dynamics, we restrict the type of questions and physical situations that can be considered. However, this brings the advantage of analyzing the paths directly in real space and not intermediated by time. In this case, the phase $\phi$ is related with the *reduced action* $S_R$ (Goldstein et al., 2013) by means of

$$\phi = \frac{S_R}{\hbar} = \frac{1}{\hbar}\int p(s)\, ds \;\; \rightarrow \;\; \phi = \frac{m\langle v\rangle L}{\hbar} \tag{5}$$

The arrow in (5) indicates a didactic adaptation where we use the mean values along the path to avoid the integral. Here L is the length of the path, $\hbar$ is the Plank constant $h$ divided by $2\,\pi$, $p = mv = \sqrt{2m(E-V)}$ is the non-relativistic momentum and the integral is along the path in coordinate space from initial point (i) to final point (f). Note that for our cases of interest the potential $V=0$.

The Eq. (5) explicitly shows the relationship between phase and path length for the electrons (L) and the matter we will use here, which completes the rules (I-IV) and makes them operative.

By means of these rules, one can explore the pattern of electron diffraction. In this case, we ask about the time independent probability that an electron emitted in the source with a fixed energy is detected in a specific place of the screen, assuming the condition of Frauenhofer. By combining (1) and (5) we obtain

$$\phi = \frac{m\,v}{\hbar} d \sin\theta \tag{6}$$

which, together with Eq. (3) and Eq. (4) bring the relations among all relevant system variables. For didactic purposes, it may not be convenient to replace (6) in (4) directly, but proceed to a stepwise analysis, where the dependency on $\phi$ is first addressed (as was done in the previous section) and then the form in that $\phi$ depends on the geometry and magnitudes associated with the electron (5). In this exploration, the use of the geometric character of the sums is promoted, which reveals an emerging wave character for the electrons. For example, the condition of the first minimum previously analyzed $N\,\phi = 2\pi$ translates into

$$(D + d)\sin\theta = \frac{h}{mv} \tag{7}$$

where $D = (N\text{-}1)\; d$ is the slit width. For $D \gg d$ the Eq. (7) has a simple geometric interpretation. It means that when the direction $\theta$ is such that the difference of lengths between the paths at both ends of the slit maintains a certain relation a minimum in the curve of probability function occurs in that direction. This implies that the vectors associated with each of the *N* paths make a complete turn and are completely canceled as shown in the inset (c) of Figure 4.

In addition, the right side of Eq. (7) reveals that this relation, which controls the position of the first minimum of the pattern depends only on the electron properties. This *characteristic* length is the De Broglie wavelength

$$\lambda = \frac{h}{m\,v} \tag{8}$$

which emerges naturally and not imposed as it is usually presented in elementary discussions of quantum mechanics. From a didactic perspective at high school level, it would not be so significant to identify with a name at this length but the role that mass and speed play in the pattern of diffraction.

Another advantage of this approach is that it allows the students to make an analysis of the transition between quantum and classical behavior. This can be done by analyzing under which geometric conditions electron diffraction is possible, or by increasing the masses or velocities in a fixed setup. In any case, it is always the interplay between the De Broglie wavelength involved and the slit size, which controls the diffraction pattern. In this time independent scheme, the shortest path emerges in the context of the Maupertuis's least action principle for real space trajectories that conserve energy (Peskin and Schroeder, 2015). The treatment of the least action principle is not considered in detail in this proposal, but it could be addressed at basic levels of the University, where students are already familiar with the concept of classic action

## LIGHT DIFFRACTION

The quantization of the electromagnetic field including its sources is a standard and sophisticated technique that describes the interaction of radiation and matter in the framework of quantum QED electrodynamics (Peskin and Schroeder, 2015). For didactic purposes, such machinery has no sense here. The quantum treatment that we consider adapts and simplifies the mathematics allowing to approach eventually all the optics from a quantum viewpoint.
The first transformation consists on using the scalar wave equation for the electromagnetic field (Born and Wolf, 1999), which rules out effects such as polarization of light not relevant in this case. The second adaptation is to consider the time independent scalar field, which is represented by the Helmholtz equation, whose quantization via path integrals gives rise to a *time independent path integral* Sawant et al. (2014). In this framework, the calculation does not consider higher energy corrections (Peskin and Schroeder, 2015), which does not play a significant role in low energy quantum optics. Under these conditions the phase $\phi$ (which complete rule III above) is given by,

$$\phi = \int k(s)\, ds \rightarrow \langle k \rangle L \tag{9}$$

The Eq. (9), despite all the approaches involved, maintains two notable didactic aspects, simplicity and predictive power. Simplicity, since it synthesizes in a unique constant $\langle k \rangle$ both properties of the light, and the medium. In fact, in vacuum, the constant $k$ depends only on the light and didactically it is enough to say that it takes certain value for each 'color' of light.

On the other hand, the transition to the classical limit emerges naturally when $k$ becomes very large. In this case, the only path that contributes is the shortest, recovering the Fermat principle of geometrical optics. Snell's law, where refraction index $n(s)$ is encapsulated in $k(s)$ is a direct consequence of this procedure. This transition plays a similar role that the emergence of the least action principle for matter mentioned above. The graphical methods to add up vectors considered in the previous sections are fully adequate to explore this limiting case.

The diffraction effects arise in our model by means of the superposition of classical paths through the same mathematics and geometric construction as in the case of electrons. The whole analysis of previous sections can be here directly applied in the context of light. The Equations (4) -(5) -(6) and (9) encapsulates the physics of quantum light diffraction.

## DISCUSSION AND CONCLUSIONS

In this work, we have analyzed the matter and light diffraction from a unified quantum viewpoint. To this end, we have employed a time independent adaptation of the Feynman's path integrals formulation, which is accessible to high school students and involves only basic operations with vectors.

The application to diffraction offers the possibility to explore the singular probabilistic character of quantum mechanics, as well as other aspects such as superposition of "classical" real space paths, and the role Plank's constant.

Although the mathematics developed here has much in common with the textbook treatment of wave diffraction (Born and Wolf, 1999) the focus and the physical interpretation is very different. On the one hand, the traditional treatment of the subject is from a classical point of view, i.e. the analysis focusses on intensity (and not probability) distributions. In addition, the wavelength concept plays a significant prior role in this type of discussions. Finally, they are mainly restricted to the wave optics. On the contrary, we propose a quantum treatment from the beginning, as indicated by the quantum rules (I-IV). Furthermore, the quantum approach is non-traditional, in the sense that quantum arguments usually begin by introducing the particle wave duality by means of the De Broglie wavelength. Here, there is no wavelength introduced a priori, but the wave properties naturally emerge from the treatment. In fact, there is no mention of any wave, particle or 'duality' character in the approach.

Not only is the language in the classical and quantum context different, but there are aspects that quantum treatment can describe, which are impossible in a classical wave context. The most evident is the formation of the diffraction pattern as a product of localized events on the screen. A classical wave model cannot account for this process, whereas the probabilistic character of the quantum model describes this naturally.

Another aspect without classical counterpart is the role of non-classical paths in the pattern formation. In our treatment, the superposition of classical paths is 'sufficient' to account for the diffraction, which is very satisfactory from a didactic point of view. However, the all paths addition prescription leads to quantum corrections to classical Fresnel theory, due to non-classical contributions, for example paths that pass several times through one or several slits (Sawant et al., 2014). Beyond the quantitative value of these corrections, the exploration of non-classical paths interference may have didactic value. Strategies of this kind could help to disrupt the natural tendency of students to think that these paths are followed by electrons, instead of a calculation procedure to obtain a probability.

Based on the didactic transposition presented here we are developing a didactic sequence to teach diffraction from a quantum and unified viewpoint. The sequence is organized in a series of situations in the form of questions and visual representations that promote the emergence of the basic inherent concepts. For this, we use graphics software tools and spreadsheets, which enhance the geometric character of the Feynman method. Subsequently the sequence will be tested and implemented in actual high school courses. The long-term goal is to generate a didactic product adaptable to different educational levels and contexts and that can be integrated with other didactic sequences that we have developed.

Finally, we would like to mention that the quantum approach to diffraction considered here is a natural playground to teach the uncertainty principle. This non-traditional approximation to the subject via path integrals is a promising didactic route, which bypasses the usual quantum wave-mechanics approach, and clearly deserves further investigation.

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